# BORCHERDS FORMS AND GENERALIZATIONS OF SINGULAR MODULI

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## 1. Introduction

Borcherds forms are meromorphic modular forms for arithmetic subgroups $\Gamma$ of the orthogonal group $O(n,2)$ which arise from a regularized theta lift of (vector valued) modular forms of weight $1-\frac{n}{2}$ for $SL_2(\mathbb{Z})$ with poles at the cusp. They have interesting product expansions and explicitly known divisors (cf. [2]). In some cases they can be realized as classical modular forms, such as the difference of two modular $j$-functions or as the discriminant function $\Delta$ (see [3]). In this paper, we give a factorization of values of Borcherds forms at CM points. The main result can be viewed as a generalization of the singular moduli result (Theorem 1.3 of [9]) of Gross and Zagier. In fact, our method gives a new proof of their result, which will be discussed in a sequel to this paper.

Let $V$ be a vector space with quadratic form $Q$ of signature $(n,2)$ and let $D$ be the space of oriented negative-definite two-planes in $V(\mathbb{R})$. $D$ is the symmetric space for $O(n,2)$ and has a Hermitian structure. For example, when $n=1$, $D\simeq\mathfrak{H}^+\sqcup\mathfrak{H}^-$ is the union of the upper and lower half-planes $\mathfrak{H}=\mathfrak{H}^+$ and $\mathfrak{H}^-$, respectively. Let $H=\mathrm{GSpin}(V)$ and let $K\subset H(\mathbb{A}_f)$ be a compact open subgroup, where $\mathbb{A}_f$ is the finite adeles. We consider the quasi-projective variety

$$X_K=H(\mathbb{Q})\backslash\Big(D\times H(\mathbb{A}_f)/K\Big)\simeq\coprod_j\Gamma_j\backslash D^+,$$

for a finite number of arithmetic subgroups $\Gamma_j\subset H(\mathbb{Q})$, and where $D^+\subset D$ is the subset of positively oriented two-planes.

Recall the theory of Borcherds forms on $X_K$. For a lattice $L\subset V$ with dual

$$L^\vee=\{x\in V\mid (x,L)\subseteq\mathbb{Z}\}$$

such that $L^\vee\supset L$, there exists a finite dimensional subspace $S_L\subset S(V(\mathbb{A}_f))$ of the Schwartz space of $V(\mathbb{A}_f)$ defined as follows. Let $\hat{L}=L\otimes_{\mathbb{Z}}\hat{\mathbb{Z}}$. Then $S_L$ is the space of functions with support in $\hat{L}^\vee$ which are constant on cosets of $\hat{L}$. A natural basis of $S_L$ is

$$\{\varphi_\eta=\mathrm{char}(\eta+L)\mid \eta\in L^\vee/L\}$$

and $\dim S_L=|L^\vee/L|$. There exists a representation $\omega$ of (the metaplectic extension $\Gamma'$ of) $\Gamma=SL_2(\mathbb{Z})$ on $S(V(\mathbb{A}_f))$ preserving $S_L$; see section 4 of [2] for details.

A modular form $F:\mathfrak{H}\to S_L$ of weight $1-\frac{n}{2}$ and type $\omega$ for $\Gamma$ satisfies

$$F(\gamma\tau)=(c\tau+d)^{1-\frac{n}{2}}\omega(\gamma)(F(\tau))$$

for all $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma$. The function $F$ has Fourier expansion

$$F(\tau)=\sum_\eta F_\eta(\tau)\varphi_\eta=\sum_\eta\sum_m c_\eta(m)\mathbf{q}^m\varphi_\eta, \tag{1}$$

where $\mathbf{q} = e^{2\pi i\tau}$. We say that $F$ is *weakly holomorphic* if only a finite number of the $c_\eta(m)$'s with $m < 0$ are non-zero. Furthermore, we call such a modular form *integral* if the non-positive Fourier coefficients lie in $\mathbb{Z}$.

To a weakly holomorphic integral modular form $F$ of weight $1 - \frac{n}{2}$, Borcherds attaches a function $\Psi(F)$ (called a Borcherds form), which is a meromorphic modular form on the space $D \times H(\mathbb{A}_f)$ with respect to $H(\mathbb{Q})$. The weight of $\Psi(F)$ is $\frac{1}{2}c_0(0)$ and the divisor of $\Psi(F)^2$ is given explicitly in terms of the negative Fourier coefficients of $F$,

$$\operatorname{div}(\Psi(F)^2) = \sum_\eta \sum_{m>0} c_\eta(-m) Z(m, \eta, K),$$

for divisors $Z(m, \eta, K)$ on $X_K$. The concrete connection between $F$ and $\Psi(F)$ is given by a regularized theta lift

$$\Phi(z, h; F) := \int_{\Gamma\backslash\mathfrak{H}}^{\bullet} ((\ F(\tau), \theta(\tau, z, h)\ )) v^{-2} du dv,$$

where $z \in D, h \in H(\mathbb{A}_f)$ and $\tau = u + iv \in \mathfrak{H}$, and where $((\ F(\tau), \theta(\tau, z, h)\ ))$ is a theta function constructed from the Fourier expansion of $F$; see section 2.1 for details. Since $F$ has a pole at the cusp, this integral diverges and so it must be regularized. See [2] or section 2.1 for the exact definition of the regularized integral. When $z$ is not in the divisor of $\Psi(F)$ we have

$$\Phi(z, h; F) = -2 \log ||\Psi(z, h; F)||^2, \tag{2}$$

where $|| \cdot ||$ is the Petersson norm, suitably normalized. Our goal is to evaluate the averages of $\Phi(F)$ over certain sets of CM points.

To define CM points, we consider a rational splitting

$$V = V_+ \oplus U,$$

where $V_+$ has signature $(n, 0)$ and $U$ has signature $(0, 2)$. This determines a two-point subset $D_0 \subset D$ consisting of $U(\mathbb{R})$ with its two possible orientations. Let $T = \mathrm{GSpin}(U)$ and $K_T = K \cap T(\mathbb{A}_f)$. We obtain a zero cycle

$$Z(U)_K = T(\mathbb{Q})\backslash\Big(D_0 \times T(\mathbb{A}_f)/K_T\Big) \hookrightarrow X_K,$$

which we regard as a set of CM points inside of $X_K$. The main theorem is

**Theorem 1.1.** *(i) $\Phi(F)$ is finite at all CM points.*

*(ii) There exist explicit constants $\kappa_\eta(m)$ such that*

$$\sum_{z \in Z(U)_K} \Phi(z; F) = \frac{4}{\mathrm{vol}(K_T)} \sum_\eta \sum_{m \geq 0} c_\eta(-m) \kappa_\eta(m), \tag{3}$$

*where the $c_\eta(-m)$'s are the negative Fourier coefficients of $F$.*

Now using relation (2) we obtain

**Corollary 1.2.** *When $Z(U)_K$ does not meet the divisor of $\Psi(F)$, we have*

$$\sum_{z \in Z(U)_K} \log ||\Psi(z; F)||^2 = \frac{-2}{\mathrm{vol}(K_T)} \sum_\eta \sum_{m \geq 0} c_\eta(-m) \kappa_\eta(m). \tag{4}$$

When $Z(U)_K$ meets the divisor of $\Psi(F)$, it remains to give an interpretation of Theorem 1.1 in terms of the function $\Psi(F)$.

The constants $\kappa_\eta(m)$ come from Eisenstein series on $SL_2$. The quantity in the left hand side of (3) can be written as an integral

$$\int_{\mathbb{S}(U)} \Phi(z_0, h; F) dh = \int_{\mathbb{S}(U)} \int^{\bullet}_{\Gamma\backslash\mathfrak{H}} (( \; F(\tau), \theta(\tau, z_0, h) \; )) v^{-2} du dv dh,$$

where $\mathbb{S}(U) = SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)$ and $z_0 \in D_0$. Here we write the theta function as a tensor product

$$\theta(\tau, z_0, h) = \theta_+(\tau, z_0) \otimes \theta_-(\tau, z_0, h)$$

of the theta functions for $V_+$ and $U$, respectively. Then we use the contraction map $\langle \cdot, \theta_+ \rangle$ (see section 3.2 for details) and write

$$(( \; F(\tau), \theta(\tau, z_0, h) \; )) = (( \; \langle F, \theta_+ \rangle(\tau), \theta_-(\tau, z_0, h) \; )),$$

where $\langle F, \theta_+ \rangle \in S(U(\mathbb{A}_f))$. After some justification, the order of integration (where the inside integral is regularized) can be switched giving

$$\int^{\bullet}_{\Gamma\backslash\mathfrak{H}} (( \; \langle F, \theta_+ \rangle(\tau), \int_{\mathbb{S}(U)} \theta_-(\tau, z_0, h) dh \; )) v^{-2} du dv. \tag{5}$$

Then by the Siegel-Weil formula, the integral of $\theta_-(\tau, z_0, h)$ on $\mathbb{S}(U)$ gives rise to a *coherent* Eisenstein series, $E(\tau, s; -1)$, of weight $-1$. For the definition of the term coherent, see [8]. With this Eisenstien series, we can write (5) as

$$\int^{\bullet}_{\Gamma\backslash\mathfrak{H}} (( \; \langle F, \theta_+ \rangle(\tau), E(\tau, 0; -1) \; )) v^{-2} du dv. \tag{6}$$

Using Maass operators we relate $E(\tau, s; -1)$ to another Eisenstein series, $E(\tau, s; +1)$, of weight $+1$ via

$$E(\tau, s; -1) v^{-2} = \frac{-4i}{s} \frac{\partial}{\partial \bar{\tau}} \{ E(\tau, s; +1) \} .$$

One phenomenon that is very specific to the case of signature $(0, 2)$ is that the resulting Eisenstein series $E(\tau, s; +1)$ is incoherent. Hence, $E(\tau, s; +1)$ satisfies an odd functional equation with respect to $s \mapsto -s$, and, therefore, vanishes at $s = 0$. The integral (6) can be evaluated using a Stokes' Theorem argument and some convergence estimates about the Fourier coefficients of $E(\tau, s; +1)$. This leads to the constants $\kappa_\eta(m)$ as follows.

For $V = V_+ \oplus U$ and $L \subset V$, let $L_+ = V_+ \cap L$ and $L_- = U \cap L$. If $\mu \in L_-^\vee / L_-$ and $\varphi_\mu = \mathrm{char}(\mu + L_-)$ we write

$$E(\tau, s; \varphi_\mu, +1) = \sum_m A_\mu(s, m, v) \mathbf{q}^m,$$

where the Fourier coefficients have Laurent expansions

$$A_\mu(s, m, v) = b_\mu(m, v) s + O(s^2).$$

In order to define $\kappa_\eta(m)$, we first define

$$\kappa^U_\mu(m) = \begin{cases} \lim_{v\to\infty} b_\mu(m, v) & \text{if } m > 0, \\ k_0(0)\varphi_\mu(0) & \text{if } m = 0, \\ 0 & \text{if } m < 0, \end{cases}$$

where $k_0(0)$ is a constant which depends on the space $U$ (see Definition 2.17). Let

$$L^\vee = \bigcup_\eta (\eta + L), \; L = \bigcup_\lambda (\lambda + L_+ + L_-)$$

and write $\eta = \eta_+ + \eta_-, \lambda = \lambda_+ + \lambda_-$. Then we define

$$\kappa_\eta(m) = \sum_\lambda \sum_{x \in \eta_+ + \lambda_+ + L_+} \kappa^U_{\eta_- + \lambda_-}(m - Q(x)).$$

The space $U$ is a rational quadratic space of signature $(0,2)$, so $U \simeq k$ for an imaginary quadratic field $k$ and the quadratic form is just a negative multiple of the norm-form. When $k$ has odd discriminant and $m \neq 0$, then

$$\frac{-2}{\mathrm{vol}(K_T)} \kappa_\eta(m)$$

is the logarithm of an integer. Thus, if $F$ has $c_0(0) = 0$, so that $\Psi(F)$ is a meromorphic function, then Corollary 1.2 shows that

$$\prod_{z \in Z(U)_K} ||\Psi(z; F)||^2 \tag{7}$$

is a rational number. Moreover, if all of the negative Fourier coefficients of $F$ are non-negative, then (7) is actually an integer. In the case of signature $(2,2)$, similar results were obtained in [5] for certain rational functions and CM points on a Hilbert modular surface. If $c_0(0) \neq 0$, then there is a transcendental factor

$$(4\pi d)^{-1}\, e^{2\frac{L'(0,\chi)}{L(0,\chi)}}$$

appearing in (7), which is related to Shimura's period invariant, [15],[8],[18], for the CM points in the 0-cycle $Z(U)_K$. This factor arises from the trivialization over the CM cycle of the line bundle of which $\Psi(F)$ defines a section.

We can say a little bit more about the rational number appearing in (7). The formulas we obtain for $\kappa_\eta(m)$ tell us the explicit factorization of the rational part of (7). Then, as a consequence of Corollary 1.2, we are able to state a Gross-Zagier type of theorem about which primes can occur in the factorization. For $F$ as in (1), define

$$m_{\max} = \max\{m > 0 \mid c_\eta(-m) \neq 0 \text{ for some } \eta\}.$$

**Theorem 1.3.** *Let $-d$ be an odd fundamental discriminant and assume $U \simeq k = \mathbb{Q}(\sqrt{-d})$. Also assume that $L_- \simeq \mathfrak{a}$ for an $\mathcal{O}_k$-ideal $\mathfrak{a}$. Then the only primes which occur in the factorization of the rational part of*

$$\prod_{z \in Z(U)_K} ||\Psi(z; F)||^2$$

*are*

*(i) $q$ such that $q \mid d$,*

*(ii) $p$ inert in $k$ with $p \leq d m_{\max}$.*

As mentioned in Theorem 1.1, one striking phenomenon that occurs in this paper is that the regularized theta lift $\Phi(F)$ is always finite! This is interesting since the Borcherds form $\Psi(F)$ can have zeroes or poles, and (2) only holds when the right

hand side is finite. Considering this, one might say that the theta lift is *over-regularized*, and it would be interesting to find the analog of Corollary 1.2 when $Z(U)_K$ meets the divisor of $\Psi(F)$.

There exists lots of recent work on singular moduli, particularly traces of singular moduli (e.g. [1], [4] and [19]). By considering the case of signature $(1,2)$, Theorem 1.3 of [9] can be recovered from Theorem 1.1. The appropriate quadratic space is

$$V = \{x \in M_2(\mathbb{Z}) \mid \mathrm{tr}(x) = 0\}$$

with $Q(x) = \det(x)$. For a particular choice of $F$,

$$\prod_{z \in Z(U)_K} \Psi(z; F) = \prod_{[\tau_1],[\tau_2]} \Big(j(\tau_1) - j(\tau_2)\Big),$$

where $\tau_1$ and $\tau_2$ are CM points with relatively prime fundamental discriminants and $[\tau_i]$ denotes an equivalence class modulo $SL_2(\mathbb{Z})$. The right hand side of (4) then gives the same factorization as in [9]. We will discuss this new proof of Gross-Zagier in a subsequent paper.

## 2. Main theorem in the case of signature $(0,2)$

2.1. **Basic Setup.** We begin by introducing some notation and relevant background material, and we refer the reader to section 1 of [12] for more details. Let $V$ be a vector space over $\mathbb{Q}$ of dimension $n+2$ with quadratic form $Q$, of signature $(n,2)$, on $V$. Let $D$ be the space of oriented negative-definite 2-planes in $V(\mathbb{R})$. For $z \in D$, let $\mathrm{pr}_z : V(\mathbb{R}) \to z$ be the projection map and, for $x \in V(\mathbb{R})$, let $R(x,z) = -(\mathrm{pr}_z(x), \mathrm{pr}_z(x))$. Then we define

$$(x,x)_z = (x,x) + 2R(x,z),$$

and our Gaussian for $V$ is the function

$$\varphi_\infty(x,z) = e^{-\pi(x,x)_z}.$$

For $\tau \in \mathfrak{H}, \tau = u + iv$, let

$$g_\tau = \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} v^{\frac{1}{2}} & \\ & v^{-\frac{1}{2}} \end{pmatrix},$$

and $g'_\tau = (g_\tau, 1) \in Mp_2(\mathbb{R})$. Let $l = \frac{n}{2} - 1$, $G = SL_2$ and $\omega$ be the Weil representation of the metaplectic group $G'_\mathbb{A}$ on $S(V(\mathbb{A}))$, the Schwartz space of $V(\mathbb{A})$. If $H = \mathrm{GSpin}(V)$, then for the linear action of $H(\mathbb{A}_f)$ we write $\omega(h)\varphi(x) = \varphi(h^{-1}x)$ for $\varphi \in S(V(\mathbb{A}_f))$. If $z \in D$ and $h \in H(\mathbb{A}_f)$, we have the linear functional on $S(V(\mathbb{A}_f))$ given by

$$\varphi \longmapsto \theta(\tau, z, h; \varphi) = v^{-\frac{l}{2}} \sum_{x \in V(\mathbb{Q})} \omega(g'_\tau)(\varphi_\infty(\cdot, z) \otimes \omega(h)\varphi)(x). \tag{8}$$

Let $L \subset V$ be a lattice with dual

$$L^\vee = \{x \in V \mid (x, L) \subseteq \mathbb{Z}\}$$

and let $S_L \subset S(V(\mathbb{A}_f))$ be the space of functions with support in $\hat{L}^\vee$ and constant on cosets of $\hat{L}$, where $\hat{L} = L \otimes_\mathbb{Z} \hat{\mathbb{Z}}$. We remark that $S_L$ is finite dimensional and has a natural basis given by

$$\{\varphi_\eta = \mathrm{char}(\eta + L) \mid \eta \in L^\vee / L\}.$$

We also have

$$S(V(\mathbb{A}_f)) = \varinjlim_{L} S_L.$$

Let $\Gamma' = Mp_2(\mathbb{Z})$ be the full inverse image of $\Gamma = SL_2(\mathbb{Z}) \subset G(\mathbb{R})$ in $G'_{\mathbb{R}}$. For $F : \mathfrak{H} \to S_L$, the Fourier expansion of $F$ can be written

$$F(\tau) = \sum_{\eta} F_\eta(\tau)\varphi_\eta = \sum_{\eta}\sum_{m} c_\eta(m)\mathbf{q}^m \varphi_\eta. \tag{9}$$

**Definition 2.1.** *We say $F : \mathfrak{H} \to S_L$ is a weakly holomorphic modular form of weight $1 - \frac{n}{2}$ and type $\omega$ for $\Gamma'$ if*
*(i) $F(\gamma'\tau) = (c\tau + d)^{1-\frac{n}{2}}\omega(\gamma')(F(\tau))$ for all $\gamma' \in \Gamma'$, where $\gamma' \mapsto \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,*
*(ii) $F$ is meromorphic at the cusp, i.e., only a finite number of the $c_\eta(m)$'s with $m < 0$ are non-zero.*

Note that when $n$ is even, $\omega$ is a representation of $G_{\mathbb{A}}$ and we can just work with $\Gamma$. The Fourier expansion in (9) is essentially the Fourier expansion given in [2], where in that paper he works with group ring elements $\mathbf{e}_\eta \in \mathbb{C}[L^\vee/L]$ instead of the Schwartz functions $\varphi_\eta$. Since the theta function $\theta(\tau, z, h)$ is a linear functional and $F(\tau) \in S(V(\mathbb{A}_f))$, we can define the $\mathbb{C}$-bilinear pairing

$$(\!(\, F(\tau), \theta(\tau, z, h) \,)\!) = \theta(\tau, z, h; F(\tau)).$$

In terms of the Fourier expansion of $F$, this is

$$(\!(\, F(\tau), \theta(\tau, z, h) \,)\!) = \sum_{\eta} F_\eta(\tau)\theta(\tau, z, h; \varphi_\eta).$$

Note that as a function of $\tau$, the above pairing is $\Gamma$-invariant (with a pole at the cusp) since the weights of $\theta$ and $F$ cancel and their types are dual. Using this pairing we define

$$\Phi(z, h; F) := \int_{\Gamma\backslash\mathfrak{H}}^{\bullet} (\!(\, F(\tau), \theta(\tau, z, h) \,)\!) d\mu(\tau),$$

where $d\mu(\tau) = v^{-2}dudv$ and the integral is regularized as in [2]. The regularization is defined by

$$\int_{\Gamma\backslash\mathfrak{H}}^{\bullet} \phi(\tau)d\mu(\tau) = \underset{\sigma=0}{\mathrm{CT}}\left\{ \lim_{t\to\infty} \int_{\mathcal{F}_t} \phi(\tau)v^{-\sigma}d\mu(\tau) \right\}, \tag{10}$$

where we take the constant term in the Laurent expansion at $\sigma = 0$ of

$$\lim_{t\to\infty} \int_{\mathcal{F}_t} \phi(\tau)v^{-\sigma}d\mu(\tau),$$

defined initially for $\mathrm{Re}(\sigma)$ sufficiently large. Here $\mathcal{F}$ is the usual fundamental domain for the action of $\Gamma$ on $\mathfrak{H}$ and

$$\mathcal{F}_t = \{\tau \in \mathcal{F} \mid \mathrm{Im}(\tau) \le t\}$$

is the truncated fundamental domain.

2.2. **Borcherds Forms.** The space $D$ is a bounded symmetric domain. It can be viewed as an open subset $\mathcal{Q}_-$ of a quadric in $\mathbb{P}(V(\mathbb{C}))$. Explicitly,

$$D \simeq \mathcal{Q}_- = \{w \in V(\mathbb{C}) \mid (w,w) = 0, (w,\bar{w}) < 0\}/\mathbb{C}^\times,$$

where the explicit isomorphism is $[z_1, z_2] \mapsto w = z_1 + iz_2$ for a properly oriented basis $[z_1, z_2]$. Assume $K$ is a compact open subgroup of $H(\mathbb{A}_f)$ such that $H(\mathbb{A}) = H(\mathbb{Q})H(\mathbb{R})^+K$, where $H(\mathbb{R})^+$ is the identity component of $H(\mathbb{R})$. Define

$$X_K := H(\mathbb{Q})\backslash\big(D \times H(\mathbb{A}_f)/K\big).$$

This is the set of complex points of a quasi-projective variety rational over $\mathbb{Q}$, and if $\Gamma_K = H(\mathbb{Q}) \cap H(\mathbb{R})^+K$, then $X_K \simeq \Gamma_K\backslash D^+$, where $D^+ \subset D$ is the subset of positively oriented 2-planes.

Let $\mathcal{L}_D$ be the restriction to $D \simeq \mathcal{Q}_-$ of the tautological line bundle on $\mathbb{P}(V(\mathbb{C}))$. From this we get a holomorphic line bundle $\mathcal{L}$ on $X_K$ equipped with a natural norm, $||\cdot||_{\mathrm{nat}}$, called the Petersson norm. Assume we have

$$V(\mathbb{R}) = V_0 + \mathbb{R}e + \mathbb{R}f,$$

where $e$ and $f$ are such that $(e,f) = 1, (e,e) = 0 = (f,f)$. Then $\mathrm{sig}(V_0) = (n-1,1)$ and for the negative cone

$$\mathcal{C} = \{y \in V_0 \mid (y,y) < 0\},$$

we have

$$D \simeq \mathbb{D} := \{z \in V_0(\mathbb{C}) \mid y = \mathrm{Im}(z) \in \mathcal{C}\}.$$

The explicit isomorphism is

$$\mathbb{D} \to V(\mathbb{C}),\ z \mapsto w(z) := z + e - Q(z)f$$

composed with projection to $\mathcal{Q}_-$. The map $z \mapsto w(z)$ can be viewed as a holomorphic section of $\mathcal{L}_D$.

We now define the notion of a modular form on $D \times H(\mathbb{A}_f)$.

**Definition 2.2.** *A modular form on $D \times H(\mathbb{A}_f)$ of weight $m \in \frac{1}{2}\mathbb{Z}$ is a function $\Psi : D \times H(\mathbb{A}_f) \to \mathbb{C}$ such that*

(1) *$\Psi(z, hk) = \Psi(z, h)$ for all $k \in K$,*
(2) *$\Psi(\gamma z, \gamma h) = j(\gamma, z)^m \Psi(z, h)$ for all $\gamma \in H(\mathbb{Q})$, where $j(\gamma, z)$ is an automorphy factor.*

Meromorphic modular forms on $D \times H(\mathbb{A}_f)$ of weight $m \in \mathbb{Z}$ can be identified with meromorphic sections of $\mathcal{L}^{\otimes m}$. If $\Psi$ is such a meromorphic modular form, then the Petersson norm of the section $(z,h) \mapsto \Psi(z,h)w(z)^{\otimes m}$ associated to $\Psi$ is

$$||\Psi(z,h)||^2_{\mathrm{nat}} = |\Psi(z,h)|^2 |y|^{2m}.$$

For reasons we will see below, we renormalize $||\cdot||_{\mathrm{nat}}$ and instead work with the following norm

$$||\Psi(z,h)||^2 := ||\Psi(z,h)||^2_{\mathrm{nat}} \left(2\pi e^{\Gamma'(1)}\right)^m.$$

The "extra" constant in the metric here is related to that occuring in [14]. Borcherds proved that the regularized integral $\Phi(z,h;F)$ satisfies the equation

$$\begin{aligned} \Phi(z,h;F) &= -2\log||\Psi(z,h;F)||^2_{\mathrm{nat}} - c_0(0)(\log(2\pi) + \Gamma'(1)) \\ &= -2\log||\Psi(z,h;F)||^2 \end{aligned} \tag{11}$$

for a meromorphic modular form $\Psi(F)$ on $D \times H(\mathbb{A}_f)$ of weight $m = \frac{1}{2}c_0(0)$ when $z$ does not lie in the divisor of $\Psi(F)$.

**Remark 2.3.** In fact, $\Phi(F)$ may still be finite for $z \in D$ even if $z$ lies in the divisor of $\Psi$. This value of $\Phi(F)$ must have another meaning there.

**Definition 2.4.** *A Borcherds form $\Psi(F)$ is a meromorphic modular form on $D \times H(\mathbb{A}_f)$ which arises (via (11)) from the regularized theta lift of a modular form $F$.*

2.3. **CM Points.** Assume that we have a rational splitting

$$V = V_+ \oplus U,$$

where $V_+$ has signature $(n, 0)$ and $U$ has signature $(0, 2)$. This determines a two-point subset $\{z_0^\pm\} = D_0 \subset D$ given by $U(\mathbb{R})$ with its two orientations. For $z_0 \in D_0$, we are interested in computing the integral

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \Phi(z_0, h; F)dh. \tag{12}$$

Let $T = \mathrm{GSpin}(U)$ and note there is a natural homomorphism $T \to H$. Let $K$ be as in section 2.2 and define $K_T = K \cap T(\mathbb{A}_f)$. Consider the set of CM points

$$Z(U)_K := T(\mathbb{Q})\backslash \Big( D_0 \times T(\mathbb{A}_f)/K_T \Big) \hookrightarrow X_K.$$

We want to compute

$$\mathrm{vol}(K_T) \sum_{z \in Z(U)_K} \Phi(z; F) = -2 \int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \Phi(z_0, h; F)dh.$$

Note that after normalizing by the volume of $K_T$, this expression is independent of the choice of $K$.

2.4. **Convergence Questions and Regularization.** First we consider the case when $n = 0$ and our space $V = U$ is negative-definite. In this case, $D = D_0$, the Gaussian is $\varphi_\infty(x) = e^{\pi(x,x)}$ and the theta function is

$$\theta(\tau, z_0, h; \varphi) = v^{\frac{1}{2}} \sum_{x \in U(\mathbb{Q})} \omega(g'_\tau) e^{\pi(x,x)} \varphi(h^{-1}x), \tag{13}$$

for any $\varphi \in S(U(\mathbb{A}_f))$. When $n = 0$ and we have a lattice $L \subset U$ we write $\mu \in L^\vee/L$ and $\varphi_\mu = \mathrm{char}(\mu + L)$. Let $F(\tau)$ be a weakly holomorphic modular form of weight 1 valued in $S_L$, and let

$$F(\tau) = \sum_\mu F_\mu(\tau)\varphi_\mu = \sum_\mu \sum_{m \in \mathbb{Q}} c_\mu(m)\mathbf{q}^m \varphi_\mu, \tag{14}$$

where $\mu$ runs over $L^\vee/L$. We assume $c_\mu(m) \in \mathbb{Z}$ for $m \leq 0$. The functions $F_\mu$ are meromorphic modular forms with some real multiplier for a congruence subgroup of $SL_2(\mathbb{Z})$, and it will be very useful to know how large their Fourier coefficients can be.

**Lemma 2.5.** *Assume $m_\mu \in \mathbb{Z}$ is such that $c_\mu(m_\mu) \neq 0$ and $c_\mu(m) = 0$ for all $m < m_\mu$. Then there are constants $C$ and $C'$ such that, for $m > 0$,*

$$|c_\mu(m)| \leq C' \left( (-m_\mu + 2)(m - m_\mu)^6 + m^6 e^{C\sqrt{m}} \right),$$

*where $C$ depends on $m_\mu$ and on the multiplier and $C'$ depends on the polar part of $F_\mu$.*

*Proof.* The cusp form of weight 12, $(2\pi)^{-12}\Delta(\tau) = \mathbf{q}\prod_{n=1}^{\infty}(1-\mathbf{q}^n)^{24}$, has Fourier expansion

$$(2\pi)^{-12}\Delta(\tau) = \sum_{N=1}^{\infty} \tau(N)\mathbf{q}^N,$$

where $|\tau(N)| \le C_1 N^6$ for some constant $C_1$. Let $\tilde{\Delta}(\tau) = (2\pi)^{-12}\Delta(\tau)$. We can look at $F_\mu/\tilde{\Delta}$, which has weight $-11 = 1 - \frac{24}{2}$. If

$$F_\mu/\tilde{\Delta} = \sum_{m=m_\mu-1}^{\infty} a_\mu(m)\mathbf{q}^m,$$

then for $m > 0$, (3.38) of [12] tells us there are constants $C_2$ and $C$ such that

$$|a_\mu(m)| \le C_2 m^{-\frac{25}{4}} e^{C\sqrt{m}},$$

where $C$ depends on $m_\mu$ and on the multiplier. We have

$$\begin{aligned} F_\mu(\tau) &= \Big(\sum_{N=1}^{\infty}\tau(N)\mathbf{q}^N\Big)\Big(\sum_{m=m_\mu-1}^{\infty} a_\mu(m)\mathbf{q}^m\Big) \\ &= \sum_{N=1}^{\infty}\sum_{m=m_\mu-1}^{\infty}\tau(N)a_\mu(m)\mathbf{q}^{N+m} \\ &= \sum_{m=m_\mu}^{\infty}\left[\sum_{N=1}^{m-m_\mu+1}\tau(N)a_\mu(m-N)\right]\mathbf{q}^m. \end{aligned}$$

Then

$$\begin{aligned} |c_\mu(m)| &= \left|\sum_{N=1}^{m-m_\mu+1}\tau(N)a_\mu(m-N)\right| \\ &= \left|\sum_{N\ge m}\tau(N)a_\mu(m-N) + \sum_{0<N<m}\tau(N)a_\mu(m-N)\right| \\ &\le C_1\sum_{N=m}^{m-m_\mu+1} N^6|a_\mu(m-N)| + C_1C_2\sum_{0<N<m} N^6(m-N)^{-\frac{25}{4}}e^{C\sqrt{m-N}}. \end{aligned}$$

We know there is a constant $C_3$ such that $|a_\mu(m)| \le C_3$ for $m \in \{m_\mu, \ldots, 0\}$, and thus

$$\begin{aligned} |c_\mu(m)| &\le C_1C_3(-m_\mu+2)(m-m_\mu)^6 + C_1C_2m^6e^{C\sqrt{m}} \\ &\le C'\left((-m_\mu+2)(m-m_\mu)^6 + m^6e^{C\sqrt{m}}\right), \end{aligned}$$

for some constant $C'$. □

In the $n = 0$ case, the following over-regularization phenomenon occurs:

**Proposition 2.6.** *For $h \in H(\mathbb{A}_f)$,*

$$\Phi(z_0, h; F) = \int_{\Gamma\backslash\mathfrak{H}}^{\bullet} ((\ F(\tau), \theta(\tau, z_0, h)\ ))d\mu(\tau)$$

*is always finite.*

*Proof.* This case corresponds to signature $(2,0)$ in [2]. In Theorem 6.2 of [2], Borcherds points out that $\Phi$ is nonsingular except along a locally finite set of codimension 2 sub-Grassmannians $\lambda^{\perp}$, for some negative norm vectors $\lambda \in L$. No such vectors exist in signature $(2,0)$. For ease of the reader, we give the proof in our notation. We have

$$\int_{\Gamma\backslash\mathfrak{H}}^{\bullet} ((\, F(\tau), \theta(\tau, z_0, h)\, ))d\mu(\tau) = \underset{\sigma=0}{\mathrm{CT}}\left\{ \lim_{t\to\infty} \int_{\mathcal{F}_t} \theta(\tau, z_0, h; F)v^{-\sigma} d\mu(\tau)\right\}, \tag{15}$$

and we can write the integral on the right hand side of (15) as

$$\int_1^t \int_{-\frac{1}{2}}^{\frac{1}{2}} \theta(\tau, z_0, h; F)v^{-\sigma} d\mu(\tau) + \int_{\mathcal{F}_1} \theta(\tau, z_0, h; F)v^{-\sigma} d\mu(\tau).$$

The integral over the compact set $\mathcal{F}_1$ is finite and independent of $t$, so we just look at the first part. By [16], we have

$$\omega(g'_\tau)e^{\pi(x,x)} = v^{\frac{1}{2}} e(uQ(x))e^{2\pi vQ(x)},$$

where $e(y) = e^{2\pi i y}$. Then (13) is

$$\theta(\tau, z_0, h; \varphi) = v \sum_{x\in U(\mathbb{Q})} e(uQ(x))e^{2\pi vQ(x)}\varphi(h^{-1}x),$$

and so the integral over $\mathcal{F}_t - \mathcal{F}_1$ is

$$\sum_{\mu}\sum_{m\in\mathbb{Q}}\sum_{x\in U(\mathbb{Q})} c_\mu(m)\varphi_\mu(h^{-1}x) \int_1^t \int_{-\frac{1}{2}}^{\frac{1}{2}} e(um)e(uQ(x))e^{-2\pi vm}e^{2\pi vQ(x)}v^{-\sigma-1}dudv. \tag{16}$$

**Lemma 2.7.** *If $m + Q(x) \notin \mathbb{Z}$, then $c_\mu(m) = 0$.*

*Proof.* When we consider the transformation law for $F$, we have $F(\tau + 1) = \omega(T)(F(\tau))$. That is, for any $x \in U(\mathbb{A}_f)$,

$$\begin{aligned}\sum_\mu\sum_m c_\mu(m)\mathbf{q}^m e(m)\varphi_\mu(x) &= \omega(T)\left(\sum_\mu\sum_m c_\mu(m)\mathbf{q}^m\varphi_\mu\right)(x)\\ &= \sum_\mu\sum_m c_\mu(m)\mathbf{q}^m\omega(T)(\varphi_\mu)(x)\\ &= \sum_\mu\sum_m c_\mu(m)\mathbf{q}^m e(-Q(x))\varphi_\mu(x).\end{aligned}$$

We see $m + Q(x) \notin \mathbb{Z}$ implies $c_\mu(m) = 0$. ◻

For $m + Q(x) \in \mathbb{Z}$,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} e(um)e(uQ(x))du = \begin{cases} 1 & \text{if } m + Q(x) = 0,\\ 0 & \text{otherwise.}\end{cases}.$$

Integrating with respect to $u$ in (16) and letting $t \to \infty$ gives

$$\sum_{\mu} \sum_{\substack{m \in \mathbb{Q} \\ m \geq 0}} \sum_{\substack{x \in U(\mathbb{Q}) \\ Q(x)+m=0}} c_\mu(m) \varphi_\mu(h^{-1}x) \int_1^\infty e^{-4\pi m v} v^{-\sigma-1} dv. \tag{17}$$

We have $m \geq 0$ since $Q(x) \leq 0$. When $m = 0$, we get

$$\sum_{\mu} c_\mu(0) \varphi_\mu(0) \int_1^t v^{-\sigma-1} dv = c_0(0) \frac{1}{\sigma}(1 - t^{-\sigma}),$$

which equals zero when we take the limit as $t \to \infty$ followed by the constant term at $\sigma = 0$. For $m > 0$, (3.35) of [12] says

$$\int_0^\infty e^{-4\pi m v} v^{-\sigma-1} dv \leq C(\epsilon, \sigma) e^{-4\pi m}$$

for any $\epsilon$ with $0 < \epsilon < 4\pi m$, where the constant $C(\epsilon, \sigma)$ is uniform in any $\sigma$-halfplane and independent of $m$. Using this in (17), we have

$$C(\epsilon, \sigma) \sum_{\mu} \sum_{m > 0} c_\mu(m) e^{-4\pi m} \sum_{\substack{x \in U(\mathbb{Q}) \\ Q(x)+m=0}} \varphi_\mu(h^{-1}x),$$

which is finite by Lemma 2.5. □

2.5. **Eisenstein Series.** Here we give the basic definition of an Eisenstein series and some related theory when $V$ has signature $(n, 2)$ for $n$ even. What follows is a summary of the explanations given in [12] for $n$ even, and we refer the reader to that paper for the more general theory. Inside of $G_{\mathbb{A}}$, we have the subgroups

$$N_{\mathbb{A}} = \{n(b) \mid b \in \mathbb{A}\}, \; n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix},$$

and

$$M_{\mathbb{A}} = \{m(a) \mid a \in \mathbb{A}^\times\}, \; m(a) = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}.$$

Define the quadratic character $\chi = \chi_V$ of $\mathbb{A}^\times / \mathbb{Q}^\times$ by

$$\chi(x) = (x, -\det(V)),$$

where $\det(V) \in \mathbb{Q}^\times / (\mathbb{Q}^\times)^2$ is the determinant of the matrix for the quadratic form $Q$ on $V$. For $s \in \mathbb{C}$, let $I(s, \chi)$ be the principal series representation of $G_{\mathbb{A}}$. This space consists of smooth functions $\Phi(s)$ on $G_{\mathbb{A}}$ such that

$$\Phi(n(b)m(a)g, s) = \chi(a)|a|^{s+1} \Phi(g, s).$$

We have a $G_{\mathbb{A}}$-intertwining map

$$\lambda = \lambda_V : S(V(\mathbb{A})) \to I\left(\frac{n}{2}, \chi\right), \tag{18}$$

where $\lambda(\varphi)(g) = (\omega(g)\varphi)(0)$. If $K_\infty = SO(2)$ and $K_f = SL_2(\hat{\mathbb{Z}})$, then a section $\Phi(s) \in I(s, \chi)$ is called standard if its restriction to $K_\infty K_f$ is independent of $s$. The function $\lambda(\varphi)$ has a unique extension to a standard section $\Phi(s) \in I(s, \chi)$ such

that $\Phi\left(\frac{n}{2}\right)=\lambda(\varphi)$. We let $P=MN$ and define the Eisenstein series associated to $\Phi(s)$ by

$$E(g,s;\Phi)=\sum_{\gamma\in P_{\mathbb{Q}}\backslash G_{\mathbb{Q}}}\Phi(\gamma g,s),$$

where $G_{\mathbb{Q}}$ is identified with its image in $G_{\mathbb{A}}$. This series converges for $\mathrm{Re}(s)>1$ and has a meromorphic analytic continuation to the whole $s$-plane.

One step in proving the $(0,2)$-Theorem is to apply Maass operators to obtain a relation between two Eisenstein series. Let

$$X_{\pm}=\frac{1}{2}\begin{pmatrix}1 & \pm i\\ \pm i & -1\end{pmatrix}\in\mathfrak{sl}_2(\mathbb{C}).$$

For $r\in\mathbb{Z}$, let $\chi_r$ be the character of $K_\infty$ defined by

$$\chi_r(k_\theta)=e^{ir\theta},\ k_\theta=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\in K_\infty.$$

Let $\phi: G_{\mathbb{R}}\to\mathbb{C}$ be a smooth function of weight $l$, meaning $\phi(gk_\theta)=\chi_l(k_\theta)\phi(g)$, and let $\xi(\tau)=v^{-\frac{l}{2}}\phi(g_\tau)$ be the corresponding function on $\mathfrak{H}$. Then $X_\pm\phi$ has weight $l\pm 2$, and the corresponding function on $\mathfrak{H}$ is

$$v^{-\frac{l\pm 2}{2}}X_\pm\phi(g_\tau)=\begin{cases}\left(2i\frac{\partial\xi}{\partial\tau}+\frac{l}{v}\xi\right)(\tau) & \text{for } +,\\ -2iv^2\frac{\partial\xi}{\partial\bar\tau}(\tau) & \text{for } -.\end{cases}$$

**Lemma 2.8** (Lemma 2.7 of [12]). *Let $\Phi^r_\infty(s)\in I_\infty(s,\chi)$ be the normalized eigenvector of weight $r$ for the action of $K_\infty$. Then*

$$X_\pm\Phi^r_\infty(s)=\frac{1}{2}(s+1\pm r)\Phi^{r\pm 2}_\infty(s).$$

For $\varphi\in S(V(\mathbb{A}_f))$, let $E(g,s;\Phi^r_\infty\otimes\lambda(\varphi))$ be the Eisenstein series of weight $r$ on $G_{\mathbb{A}}$ associated to $\varphi$. For the Gaussian, $\varphi_\infty(x,z)$, we have $\lambda(\varphi_\infty)=\Phi^l_\infty\left(\frac{n}{2}\right)$, where $l=\frac{n}{2}-1$. This means that

$$X_-E(g,s;\Phi^{l+2}_\infty\otimes\lambda(\varphi))=\frac{1}{2}(s-l-1)E(g,s;\Phi^l_\infty\otimes\lambda(\varphi)).$$

On $\mathfrak{H}$, this translates to

$$-2iv^2\frac{\partial}{\partial\bar\tau}\Big\{E(\tau,s;\varphi,l+2)\Big\}=\frac{1}{2}\left(s-\frac{n}{2}\right)E(\tau,s;\varphi,l), \tag{19}$$

where we write $E(\tau,s;\varphi,l)=v^{-\frac{l}{2}}E(g_\tau,s;\Phi^l_\infty\otimes\lambda(\varphi))$. One main result we need is the Siegel-Weil formula.

**Theorem 2.9** (Siegel-Weil formula). *Let $V$ be a vector space of signature $(n,2)$. Assume $V$ is anisotropic or that $\dim(V)-r_0>2$, where $r_0$ is the Witt index of $V$. Then $E(g,s;\varphi)$ is holomorphic at $s=\frac{n}{2}$ and*

$$E\left(g,\frac{n}{2};\varphi\right)=\frac{\alpha}{2}\int_{SO(V)(\mathbb{Q})\backslash SO(V)(\mathbb{A})}\theta(g,h;\varphi)dh,$$

*where $dh$ is Tamagawa measure on $SO(V(\mathbb{A}))$, and $\alpha$ is 2 if $n=0$ and is 1 otherwise.*

Here $\theta(g,h;\varphi)$ is defined as in (8) without $v^{-\frac{l}{2}}$ and with $g$ replacing $g'_\tau$. The integration for $SO(U)(\mathbb{R})$ is with respect to the action $h_\infty^{-1}x$ in the argument of $\varphi_\infty$. The cases which are omitted in the Siegel-Weil formula are when $n=1=r_0$ ($V$ is isotropic) and $n=2=r_0$ ($V$ is split).

Let us now consider the situation $V=U$, $\mathrm{sig}(U)=(0,2)$. The representation we are interested in is $I(0,\chi)$. This global principal series is a restricted tensor product of local ones,

$$I(0,\chi)=\otimes'_v I_v(0,\chi_v).$$

For the local space $U_v=U(\mathbb{Q}_v)$, define the quadratic character $\chi_v$ of $\mathbb{Q}_v^\times$ by

$$\chi_v(x)=(x,-\det(U_v))_v.$$

Let $R_v(U)$ be the maximal quotient of $S(U_v)$ on which $O(U_v)$ acts trivially. The following proposition is a special case of Proposition 1.1 of [11].

**Proposition 2.10.** *(i) If $v\neq\infty$, then*

$$I_v(0,\chi_v)=R_v(U^+)\oplus R_v(U^-),$$

*where $U^\pm$ has Hasse invariant $\epsilon_v(U^\pm)=\pm1$.*
*(ii) If $v=\infty$, then*

$$I_\infty(0,\chi_\infty)=R_\infty(U(0,2))\oplus R_\infty(U(2,0)),$$

*and the spaces $U(0,2)$ and $U(2,0)$ have opposite Hasse invariants.*

Recall the notion of an incoherent collection.

**Definition 2.11.** *An incoherent collection $\mathcal{C}=\{\mathcal{C}_v\}$ of quadratic spaces is a set of quadratic spaces $\mathcal{C}_v$ such that*

1. *For all $v$, $\dim_{\mathbb{Q}_v}(\mathcal{C}_v)=2$, and $\chi_{\mathcal{C}_v}=\chi$.*
2. *For almost all $v$, $\mathcal{C}_v\simeq U_v$.*
3. *(Incoherence condition) The product formula fails for the Hasse invariants:*

$$\prod_v \epsilon_v(\mathcal{C}_v)=-1.$$

Then we have, cf. (2.10) in [11],

$$I(0,\chi)\simeq\left(\bigoplus_{U'}\Pi(U')\right)\oplus\left(\bigoplus_{\mathcal{C}}\Pi(\mathcal{C})\right)$$

as a sum of two irreducible pieces defined as follows. $U'$ runs over all global quadratic spaces of dimension 2 with $\chi_{U'}=\chi$, while $\mathcal{C}$ runs over all incoherent collections of dimension 2 and character $\chi$, and

$$\Pi(U')=\otimes'_v R_v(U'),\ \Pi(\mathcal{C})=\otimes'_v R_v(\mathcal{C}).$$

For $\lambda=\lambda_U$ as in (18), we have $\lambda(\varphi_\infty)=\Phi_\infty^{-1}(0)$, where $\Phi_\infty^{-1}$ is the normalized eigenvector of weight $-1$ for the action of $K_\infty$. From the theory of principal series representations, we have $\Phi_\infty^{-1}(0)\in R_\infty(U(0,2))$ and $\Phi_\infty^{1}(0)\in R_\infty(U(2,0))$. Then Lemma 2.8 implies

$$X_+\Phi_\infty^{-1}(s)=\frac{1}{2}s\Phi_\infty^{1}(s), \tag{20}$$

so we see that the Maass operator $X_+$ shifts the coherent Eisenstein series $E(g,s;\Phi_\infty^{-1}\otimes\lambda(\varphi))$ to the *incoherent* Eisenstein series $E(g,s;\Phi_\infty^{1}\otimes\lambda(\varphi))$. Theorem 2.2 of [11] then tells us that

$$E(g,0;\Phi_\infty^{1}\otimes\lambda(\varphi))=0.$$

2.6. **The** $(0,2)$**-Theorem.** The integral we want to compute is

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)}\Phi(z_0,h;F)dh, \tag{21}$$

which is equal to

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)}\int_{\Gamma\backslash\mathfrak{H}}^{\bullet}((\ F(\tau),\theta(\tau,z_0,h)\ ))d\mu(\tau)dh. \tag{22}$$

As in [12], we would like to be able to switch the order of integration, where the inside integral is regularized. That is, we want (22) to equal

$$\int_{\Gamma\backslash\mathfrak{H}}^{\bullet}((\ F(\tau),\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)}\theta(\tau,z_0,h)dh\ ))d\mu(\tau).$$

Note that $F:\mathfrak{H}\to S_L$ implies $F(\tau)\in S(U(\mathbb{A}_f))^K$, where

$$K=\{h\in H(\mathbb{A}_f)\mid h(\lambda+L)=\lambda+L,\forall\lambda\in L^\vee/L\}$$

is an open subset of $H(\mathbb{A}_f)$.

Before we justify the interchange of integrals, we need to make some remarks about our specific case. For a reference on Clifford algebras, see [6] or [10]. The Clifford algebra $C(U)$ can be written as $C(U)=C^0(U)\oplus C^1(U)$, where $C^0(U)$ and $C^1(U)$ are the even and odd parts, respectively. $C^0(U)^\times$ acts on $C^1(U)$ by conjugation. Assume $U$ has basis $\{u,v\}$ with $Q(u)=a,Q(v)=b$ and $(u,v)=0$. Then $C(U)$ is spanned by $\{1,u,v,uv\}$ with $C^0(U)=\mathrm{span}\{1,uv\}$ and $C^1(U)=\mathrm{span}\{u,v\}$. By definition,

$$H=\{g\in C^0(U)^\times\mid gUg^{-1}=U\}.$$

Since $C^1(U)=U$, $H=C^0(U)^\times$. In $C^0(U)$ we have $(uv)^2=-ab$, so if $k=\mathbb{Q}\left(\sqrt{-ab}\right)$, then $H\simeq k^\times$. This means $SO(U)\simeq k^1$ and $k^\times\to k^1$ is the map

$$x\mapsto\frac{x}{x^\sigma}$$

by Hilbert's Theorem 90. We have the exact sequence

$$1\to Z\to H\to SO(U)\to 1,$$

where $H(\mathbb{A}_f)\simeq k_{\mathbb{A}_f}^\times, H(\mathbb{Q})\simeq k^\times, Z(\mathbb{A}_f)\simeq\mathbb{Q}_{\mathbb{A}_f}^\times$ and $Z(\mathbb{Q})\simeq\mathbb{Q}^\times$.

**Lemma 2.12.** *For any negative-definite space $U$ with quadratic form $Q$ of signature $(0,2)$, we realize $U\simeq k$ for an imaginary quadratic field $k$ and $Q$ is given by a negative multiple of the norm-form.*

If $B(h)$ is a function on $H(\mathbb{A}_f)$ which only depends on the image of $h$ in $SO(U)(\mathbb{A}_f)$, then we can view $B$ as a function on $SO(U)(\mathbb{A}_f)$ as well.

**Lemma 2.13.** *Let $B(h)$ be a function on $H(\mathbb{A}_f)$ depending only on the image of $h$ in $SO(U)(\mathbb{A}_f)$. Assume $B$ is invariant under $K$ and $H(\mathbb{Q})$. Then*

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} B(h)dh = \mathrm{vol}(K) \sum_{h\in H(\mathbb{Q})\backslash H(\mathbb{A}_f)/K} B(h),$$

*and the sum is finite.*

*Proof.* We have the exact sequence

$$1 \to k_{\mathbb{A}}^0 \to k_{\mathbb{A}}^\times \to \mathbb{R}_+^\times \to 1,$$

where the map to $\mathbb{R}_+^\times$ is the absolute value map. By the product formula, $k^\times \subset k_{\mathbb{A}}^0$ and we know $k^\times\backslash k_{\mathbb{A}}^0$ is compact.

**Lemma 2.14.** $k_{\mathbb{A}}^\times = \mathbb{Q}_{\mathbb{A}}^\times k_{\mathbb{A}}^0$.

*Proof.* $\mathbb{Q}_{\mathbb{A}}^\times$ injects into $k_{\mathbb{A}}^\times$ and also maps onto $\mathbb{R}_+^\times$. So if $(a) \in k_{\mathbb{A}}^\times$ then $\exists(b) \in \mathbb{Q}_{\mathbb{A}}^\times$ with $|(b)| = |(a)|$. Then $(b) \in \mathbb{Q}_{\mathbb{A}}^\times \subset k_{\mathbb{A}}^\times$ implies $k_{\mathbb{A}}^0(b) = k_{\mathbb{A}}^0(a)$, so $(a) \in \mathbb{Q}_{\mathbb{A}}^\times k_{\mathbb{A}}^0$. □

Lemma 2.14 implies

$$k^\times\backslash k_{\mathbb{A}}^0 \twoheadrightarrow k^\times\mathbb{Q}_{\mathbb{A}}^\times\backslash k_{\mathbb{A}}^\times,$$

and so $k^\times\mathbb{Q}_{\mathbb{A}}^\times\backslash k_{\mathbb{A}}^\times$ is also compact. The set we integrate over is

$$SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f) = H(\mathbb{Q})\backslash H(\mathbb{A}_f)/Z(\mathbb{A}_f) \simeq k^\times\mathbb{Q}_{\mathbb{A}_f}^\times\backslash k_{\mathbb{A}_f}^\times.$$

This is compact since $k^\times\mathbb{Q}_{\mathbb{A}}^\times\backslash k_{\mathbb{A}}^\times$ maps onto it. Then $K$ is open and $K \supset Z(\mathbb{A}_f)$ so $H(\mathbb{Q})\backslash H(\mathbb{A}_f)/K$ is finite. The volume term appears since $B$ is $K$-invariant. □

**Proposition 2.15.**

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \int_{\Gamma\backslash\mathfrak{H}}^\bullet (( F(\tau), \theta(\tau, z_0, h) ))d\mu(\tau)dh$$
$$= \int_{\Gamma\backslash\mathfrak{H}}^\bullet (( F(\tau), \int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \theta(\tau, z_0, h)dh ))d\mu(\tau).$$

*Proof.* The main point is that since $F(\tau) \in S(U(\mathbb{A}_f))^K$, we know

$$\int_{\Gamma\backslash\mathfrak{H}}^\bullet (( F(\tau), \theta(\tau, z_0, h) ))d\mu(\tau)$$

is $K$-invariant. So if we let

$$B(h) = \int_{\Gamma\backslash\mathfrak{H}}^\bullet (( F(\tau), \theta(\tau, z_0, h) ))d\mu(\tau),$$

then Lemma 2.13 says

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} B(h)dh = \mathrm{vol}(K) \sum_{h\in H(\mathbb{Q})\backslash H(\mathbb{A}_f)/K} B(h)$$
$$= \int_{\Gamma\backslash\mathfrak{H}}^\bullet \mathrm{vol}(K) \sum_{h\in H(\mathbb{Q})\backslash H(\mathbb{A}_f)/K} \theta(\tau, z_0, h; F(\tau))d\mu(\tau), \tag{23}$$

since the sum is finite. Now apply Lemma 2.13 again to $\theta(\tau, z_0, h; F(\tau))$ and (23) is

$$= \int_{\Gamma\backslash\mathfrak{H}}^\bullet (( F(\tau), \int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \theta(\tau, z_0, h)dh ))d\mu(\tau).$$

□

The quadratic space $U$ is anisotropic, so we can apply Theorem 2.9. This tells us that for any $\varphi \in S(U(\mathbb{A}))$,

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A})} \theta(\tau, z_0, h; \varphi) dh = v^{\frac{1}{2}} E(g_\tau, 0; \varphi, -1), \tag{24}$$

where $E(g_\tau, s; \varphi, -1)$ is a coherent Eisenstein series of weight $-1$. Since $\theta(\tau, z_0, h)$ is $SO(U)(\mathbb{R})$-invariant, it suffices to integrate over $SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)$. We choose a factorization for the measure $dh = dh_\infty \times dh_f$ such that $\mathrm{vol}(SO(U)(\mathbb{R})) = 1$.

**Lemma 2.16.**

$$(i) \quad \int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \theta(\tau, z_0, h_f; \varphi) dh_f = v^{\frac{1}{2}} E(g_\tau, 0; \varphi, -1).$$

$$(ii) \quad \mathrm{vol}(K)^{-1} = \frac{1}{2}(\#(H(\mathbb{Q})\backslash H(\mathbb{A}_f)/K)).$$

We let

$$E(\tau, s; -1) := v^{\frac{1}{2}} E(g_\tau, s; -1).$$

Then for (21) we have

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \Phi(z_0, h; F) dh = \int_{\Gamma\backslash\mathfrak{H}}^{\bullet} ((\ F(\tau), E(\tau, 0; -1)\ )) d\mu(\tau). \tag{25}$$

For $F$ as in (14), the right hand side of (25) is

$$\int_{\Gamma\backslash\mathfrak{H}}^{\bullet} ((\ F(\tau), E(\tau, 0; -1)\ )) d\mu(\tau) = \int_{\Gamma\backslash\mathfrak{H}}^{\bullet} \sum_{\mu} F_\mu(\tau) E(\tau, 0; \varphi_\mu, -1) v^{-2} du dv. \tag{26}$$

Let

$$I(s,t) := \int_{\mathcal{F}_t} \sum_{\mu} F_\mu(\tau) E(\tau, s; \varphi_\mu, -1) v^{-2} du dv.$$

In order to state the main theorem of this chapter, we view $U \simeq k = \mathbb{Q}(\sqrt{-d})$, where $-d$ is the discriminant of $k$, and let $\chi_d$ be the character of $\mathbb{Q}_{\mathbb{A}}^\times$ defined by $\chi_d(x) = (x, -d)_{\mathbb{A}}$. We define the normalized $L$-series

$$\Lambda(s, \chi_d) = \pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s, \chi_d).$$

**Definition 2.17.** *For $\varphi \in S(U(\mathbb{A}_f))$, let*

$$E(\tau, s; \varphi, +1) = \sum_m A_\varphi(s, m, v) \mathbf{q}^m,$$

*where the Fourier coefficients have Laurent expansions*

$$A_\varphi(s, m, v) = b_\varphi(m, v) s + O(s^2)$$

*at $s = 0$. For any $\varphi \in S(U(\mathbb{A}_f))$, define*

$$\kappa_\varphi(m) := \begin{cases} \lim_{v\to\infty} b_\varphi(m, v) & \text{if } m > 0, \\ k_0(0)\varphi(0) & \text{if } m = 0, \\ 0 & \text{if } m < 0, \end{cases}$$

*where*

$$k_0(0) = \log(d) + 2\frac{\Lambda'(1,\chi_d)}{\Lambda(1,\chi_d)}. \tag{27}$$

For $\varphi = \varphi_\mu = \mathrm{char}(\mu + L)$ we write

$$A_\mu(s,m,v) = A_{\varphi_\mu}(s,m,v),\ b_\mu(m,v) = b_{\varphi_\mu}(m,v),\ \kappa_\mu(m) = \kappa_{\varphi_\mu}(m).$$

**Theorem 2.18** (The $(0,2)$-Theorem)**.** *Let $F : \mathfrak{H} \to S_L \subset S(U(\mathbb{A}_f))$ be a weakly holomorphic modular form for $SL_2(\mathbb{Z})$ of weight 1, with Fourier expansion*

$$F(\tau) = \sum_\mu F_\mu(\tau)\varphi_\mu = \sum_\mu \sum_m c_\mu(m)\mathbf{q}^m \varphi_\mu,$$

*where $\mu$ runs over $L^\vee/L$ for some lattice $L$. Also, assume $c_\mu(m) \in \mathbb{Z}$ for $m \le 0$. Let*

$$\Phi(z_0, h; F) = \int^\bullet_{\Gamma\backslash\mathfrak{H}} ((\ F(\tau), \theta(\tau, z_0, h)\ ))d\mu(\tau).$$

*Then*

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \Phi(z_0, h; F)dh = 2\sum_\mu \sum_{m\ge 0} c_\mu(-m)\kappa_\mu(m).$$

*Proof.* Our proof is similar to that in [12]. The integral we want to compute is given by (26). Letting $l = -1$ in (19), we have

$$E(\tau, s; \varphi_\mu, -1)v^{-2} = \frac{-4i}{s}\frac{\partial}{\partial\bar\tau}\left\{E(\tau, s; \varphi_\mu, +1)\right\}.$$

This means we can write

$$I(s,t) = \frac{1}{2i}\int_{\mathcal{F}_t} d\Big(\sum_\mu F_\mu(\tau)\frac{-4i}{s}E(\tau, s; \varphi_\mu, +1)d\tau\Big).$$

By Stokes' Theorem, this is

$$\begin{aligned}
&= \frac{-2}{s}\int_{\partial\mathcal{F}_t}\sum_\mu F_\mu(\tau)E(\tau, s; \varphi_\mu, +1)d\tau \\
&= \frac{-2}{s}\int_{\frac12+it}^{-\frac12+it}\sum_\mu F_\mu(\tau)E(\tau, s; \varphi_\mu, +1)du \\
&= \frac{2}{s}\cdot \text{const. term of } \Big(\sum_\mu F_\mu(\tau)E(\tau, s; \varphi_\mu, +1)\Big)\Big|_{v=t}. 
\end{aligned} \tag{28}$$

The definition of the regularized integral implies

$$\int^\bullet_{\Gamma\backslash\mathfrak{H}} ((\ F(\tau), E(\tau, 0)\ ))d\mu(\tau) =$$
$$\underset{\sigma=0}{\mathrm{CT}}\Big\{\lim_{t\to\infty}\int_{\mathcal{F}_t}\sum_\mu F_\mu(\tau)E(\tau, 0; \varphi_\mu, -1)v^{-\sigma-2}dudv\Big\}.$$

We need Proposition 2.5 of [12] to hold for $n = 0$. If we use Proposition 2.6 of [12] and the fact that a factor of 2 appears in the Siegel-Weil formula here, then in our notation the analogue of Proposition 2.5 of [12] is

**Proposition 2.19.**

$$\begin{aligned}&\operatorname*{CT}_{\sigma=0}\left\{\lim_{t\to\infty}\int_{\mathcal{F}_t}\sum_{\mu}F_\mu(\tau)E(\tau,0;\varphi_\mu,-1)v^{-\sigma-2}dudv\right\}\\&=\lim_{t\to\infty}\left[\int_{\mathcal{F}_t}\sum_{\mu}F_\mu(\tau)E(\tau,0;\varphi_\mu,-1)v^{-2}dudv-2c_0(0)\log(t)\right].\end{aligned}$$

*Proof.* From Lemma 2.13, the left hand side of the desired identity is

$$\operatorname{vol}(K)\sum_h\operatorname*{CT}_{\sigma=0}\left\{\lim_{t\to\infty}\int_{\mathcal{F}_t}((F(\tau),\theta(\tau,z_0,h)\ ))v^{-\sigma-2}dudv\right\},$$

where $\operatorname{vol}(K)=\frac{2}{\#(H(\mathbb{Q})\backslash H(\mathbb{A}_f)/K)}$. Fixing $h$, we have

(29)

$$\operatorname*{CT}_{\sigma=0}\left\{\lim_{t\to\infty}\int_{\mathcal{F}_t-\mathcal{F}_1}((F(\tau),\theta(\tau,z_0,h)\ ))v^{-\sigma-2}dudv\right\}+\int_{\mathcal{F}_1}((F(\tau),\theta(\tau,z_0,h)\ ))d\mu(\tau).$$

The first term in (29) can be written as

$$\operatorname*{CT}_{\sigma=0}\left\{\lim_{t\to\infty}\int_1^t C(v,h)v^{-\sigma-1}dv\right\},\tag{30}$$

where

$$\begin{aligned}C(v,h)&=v^{-1}\int_{-\frac{1}{2}}^{\frac{1}{2}}((F(\tau),\theta(\tau,z_0,h)\ ))du\\&=\text{const. term of }v^{-1}((F(\tau),\theta(\tau,z_0,h)\ ))\\&=\sum_\mu\sum_{\substack{m\in\mathbb{Q}\\m\geq0}}c_\mu(m)\sum_{\substack{x\in U(\mathbb{Q})\\Q(x)+m=0}}\varphi_\mu(h^{-1}x)e^{4\pi vQ(x)}.\end{aligned}$$

Then we write (30) as

$$\operatorname*{CT}_{\sigma=0}\left\{\lim_{t\to\infty}\int_1^t[C(v,h)-c_0(0)]v^{-\sigma-1}dv+\lim_{t\to\infty}\int_1^t c_0(0)v^{-\sigma-1}dv\right\}.\tag{31}$$

As in [12],

$$\int_1^\infty[C(v,h)-c_0(0)]v^{-\sigma-1}dv$$

is a holomorphic function of $\sigma$. Note, this fact follows, in part, from Lemma 2.5. For the other piece of (31) we have

$$\int_1^t c_0(0)v^{-\sigma-1}dv=c_0(0)\frac{1}{\sigma}(1-t^{-\sigma}).$$

This term makes no contribution when we take the limit as $t \to \infty$ followed by the constant term at $\sigma = 0$. We are left with

$$\lim_{t\to\infty}\left[\int_1^t C(v,h)v^{-1}dv - \int_1^t c_0(0)v^{-1}dv\right] = \lim_{t\to\infty}\left[\int_1^t C(v,h)v^{-1}dv - c_0(0)\log(t)\right].$$

We have the volume term in front and we sum over $h \in H(\mathbb{Q})\backslash H(\mathbb{A}_f)/K$, so this adds on a factor of 2. □

We point out that the value $c_0(0)$ appearing in (14) and in Proposition 2.19 is independent of the choice of $L$. If we view $F(\tau) \in S(U(\mathbb{A}_f))$ as $F(\tau, x)$ for $x \in U(\mathbb{A}_f)$, then $c_0(0)$ is the zeroth Fourier coefficient of $F(\tau, 0)$. Proposition 2.19 tells us that

$$\begin{aligned}
&\underset{\sigma=0}{\mathrm{CT}}\left\{\lim_{t\to\infty}\int_{\mathcal{F}_t}\sum_{\mu} F_\mu(\tau)E(\tau,0;\varphi_\mu,-1)v^{-\sigma-2}dudv\right\}\\
&= \lim_{t\to\infty}\left[\int_{\mathcal{F}_t}\sum_{\mu} F_\mu(\tau)E(\tau,0;\varphi_\mu,-1)v^{-2}dudv \;-\; 2c_0(0)\log(t)\right]\\
&= \lim_{t\to\infty}\Big[I(0,t) - 2c_0(0)\log(t)\Big].
\end{aligned}$$

We need to compute $I(0,t)$. We have

$$A_\mu(s,m,v) = b_\mu(m,v)s + O(s^2), \tag{32}$$

where there is no constant term in $A_\mu(s,m,v)$ since $E(\tau,s;\varphi_\mu,+1)$ vanishes at $s = 0$. Then (28) implies

$$I(s,t) = \frac{2}{s}\sum_{\mu}\sum_{m} c_\mu(-m)A_\mu(s,m,t),$$

so using (32) we have

$$I(0,t) = 2\sum_{\mu}\sum_{m} c_\mu(-m)b_\mu(m,t). \tag{33}$$

Now we show that parts (i) and (ii) of Proposition 2.11 of [12] hold for $n = 0$.

**Proposition 2.20.** *(i) For $m < 0$, $b_\mu(m,t)$ decays exponentially as $t \to \infty$.*
*(ii)*

$$\lim_{t\to\infty}\left(2\sum_{\mu}\sum_{m<0} c_\mu(-m)b_\mu(m,t)\right) = 0.$$

*Proof.* If $\varphi_\mu = \otimes_p \varphi_{\mu,p} \in S(U(\mathbb{A}_f))$ and

$$E(\tau,s;\varphi_\mu,+1) = \sum_{m} E_m(\tau,s;\varphi_\mu,+1),$$

then for $m \neq 0$ we have the product formula

$$E_m(\tau,s;\varphi_\mu,+1) = A_\mu(s,m,v)\mathbf{q}^m = W_{m,\infty}(\tau,s;+1)\prod_p W_{m,p}(s,\varphi_{\mu,p}),$$

where $W_{m,\infty}(\tau,s;+1)$ and $W_{m,p}(s,\varphi_{\mu,p})$ are the local Whittaker factors at $\infty$ and $p$, respectively. Proposition 2.6 (iii) of [13] tells us that for $m < 0$,

$$W_{m,\infty}(\tau,0;+1) = 0,$$

and

$$W'_{m,\infty}(\tau,0;+1) = \pi i \mathbf{q}^m \int_1^\infty r^{-1} e^{-4\pi|m|vr} dr.$$

For the finite primes we have

$$C(m) := \left(\prod_p W_{m,p}(s,\varphi_{\mu,p})\right)\Big|_{s=0} = O(1).$$

Then

$$\begin{aligned} b_\mu(m,t) &= C(m) W'_{m,\infty}(\tau,0;+1) \\ &= C(m)\pi i \mathbf{q}^m \int_1^\infty r^{-1} e^{-4\pi|m|vr} dr, \end{aligned}$$

and we have

$$|b_\mu(m,t)| = O\Big(v^{-1}|m|^{-1}e^{-4\pi|m|v}\Big).$$

This proves (i). Part (ii) then follows from Lemma 2.5. □

Part (ii) of Proposition 2.20 tells us that we may ignore the sum on $m<0$ in (33). This means our formula for the integral is

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \Phi(z_0,h;F)dh \ = \lim_{t\to\infty}\left[2\sum_\mu \sum_{m\geq 0} c_\mu(-m) b_\mu(m,t) - 2c_0(0)\log(t)\right].$$

We can improve this by looking at the $m=0$ part. The analogue of Proposition 2.11 (iii) of [12] is

**Lemma 2.21.** *For $m=0$,*

$$b_0(0,t) - \log(t) = \log(d) + 2\frac{\Lambda'(1,\chi_d)}{\Lambda(1,\chi_d)},$$

*and for $\mu \neq 0, b_\mu(0,t) = 0$.*

*Proof.* By Theorem 3.1 of [17], we have

$$\begin{aligned} E_0(\tau,s;\varphi_\mu,+1) &= v^{\frac{s}{2}}\varphi_\mu(0) + W_{0,\infty}(\tau,s;+1)\prod_p W_{0,p}(s,\varphi_{\mu,p}) \\ &= v^{\frac{s}{2}}\varphi_\mu(0) - 2\pi i \frac{2^{-s}\Gamma(s)v^{-\frac{s}{2}}}{\Gamma\left(\frac{s}{2}+1\right)\Gamma\left(\frac{s}{2}\right)}\prod_p W_{0,p}(s,\varphi_{\mu,p}), \end{aligned}$$

which by the duplication formula is

$$= v^{\frac{s}{2}}\varphi_\mu(0) - \sqrt{\pi} i v^{-\frac{s}{2}} \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)}\prod_p W_{0,p}(s,\varphi_{\mu,p}).$$

Theorem 5.2 of [17] implies $W_{0,p}(s,\varphi_{\mu,p})=0$ if $\varphi_{\mu,p}$ is not the characteristic function of the local lattice. So $b_\mu(0,t)=0$ for $\mu\neq 0$. Now let $\mu=0$. Propositions 2.1 and 6.3 of [17] imply

$$E_0(\tau,s;\varphi_0,+1)=v^{\frac{s}{2}}-\sqrt{\pi}v^{-\frac{s}{2}}\frac{\Gamma\left(\frac{s+1}{2}\right)L(s,\chi_d)}{\Gamma\left(\frac{s}{2}+1\right)L(s+1,\chi_d)}\mathcal{C}_0,$$

where

$$\mathcal{C}_0=2^{\beta_2}\prod_{\substack{q\mid d\\ q=\text{odd prime}}}q^{-\frac{1}{2}}$$

and

$$\beta_2=\begin{cases}0 & \text{if 2 is unramified,}\\ -1 & \text{if } 4\mid d \text{ and } 8\nmid d,\\ -\frac{3}{2} & \text{if } 8\mid d.\end{cases}$$

Then $\mathcal{C}_0=d^{-\frac{1}{2}}$. We have

$$\begin{aligned}E_0(\tau,s;\varphi_0,+1)&=v^{\frac{s}{2}}-v^{-\frac{s}{2}}\frac{\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)L(s,\chi_d)}{\pi^{-\frac{s}{2}-1}\Gamma\left(\frac{s}{2}+1\right)L(s+1,\chi_d)}d^{-\frac{1}{2}}\\ &=v^{\frac{s}{2}}-v^{-\frac{s}{2}}\frac{\Lambda(s,\chi_d)}{\Lambda(s+1,\chi_d)}d^{-\frac{1}{2}}.\end{aligned}$$

The functional equation for $\Lambda(s,\chi_d)$ (cf. [7]) is

$$\Lambda(s,\chi_d)=d^{\frac{1}{2}-s}\Lambda(1-s,\chi_d).$$

We normalize $E_0(\tau,s;\varphi_0,+1)$ by $d^{\frac{s+1}{2}}\Lambda(s+1,\chi_d)$ giving

$$\begin{aligned}E_0^*(\tau,s;\varphi_0,+1)&=d^{\frac{s+1}{2}}v^{\frac{s}{2}}\Lambda(1+s,\chi_d)-d^{\frac{s+1}{2}}v^{-\frac{s}{2}}d^{-s}\Lambda(1-s,\chi_d)\\ &=d^{\frac{s+1}{2}}v^{\frac{s}{2}}\Lambda(1+s,\chi_d)-d^{\frac{1-s}{2}}v^{-\frac{s}{2}}\Lambda(1-s,\chi_d).\end{aligned}$$

Hence,

$$\begin{aligned}E_0^{*,\prime}(\tau,0;\varphi_0,+1)&=2\frac{\partial}{\partial s}\left\{d^{\frac{s+1}{2}}v^{\frac{s}{2}}\Lambda(1+s,\chi_d)\right\}\Big|_{s=0}\\ &=d^{\frac{1}{2}}\Lambda(1,\chi_d)\left\{\log(d)+\log(v)+2\frac{\Lambda'(1,\chi_d)}{\Lambda(1,\chi_d)}\right\}\\ &=h_k\left\{\log(d)+\log(v)+2\frac{\Lambda'(1,\chi_d)}{\Lambda(1,\chi_d)}\right\},\end{aligned}$$

by the residue formula. Then since $E^{*,\prime}(\tau,0;\varphi_0,+1)=h_kE'(\tau,0;\varphi_0,+1)$, we have

$$b_0(0,t)-\log(t)=\log(d)+2\frac{\Lambda'(1,\chi_d)}{\Lambda(1,\chi_d)}.$$

□

Now the $m=0$ part is

$$2\sum_{\mu}c_\mu(0)b_\mu(0,t)-2c_0(0)\log(t)=2\sum_{\mu\neq 0}c_\mu(0)b_\mu(0,t)+2c_0(0)(b_0(0,t)-\log(t)),$$

and Lemma 2.21 tells us that this expression is $2c_0(0)k_0(0)$. This finishes the proof of Theorem 2.18. □

## 3. Main theorem in the case of signature $(n, 2)$

3.1. **The Rational Splitting $V = V_+ \oplus U$.** Now we consider the general case. Assume that we have a decomposition $V = V_+ \oplus U$ where $V_+$ has signature $(n, 0)$ and $U$ has signature $(0, 2)$. For $x \in V$, write $x = x_1 + x_2$, $x_1 \in V_+, x_2 \in U$. Let $z_0 \in D_0$. Then $R(x, z_0) = -(x_2, x_2)$ so we see

$$\varphi_\infty(x, z_0) = e^{-\pi(x,x)_{z_0}} = e^{-\pi[(x_1,x_1)-(x_2,x_2)]} = e^{-\pi(x_1,x_1)}e^{\pi(x_2,x_2)},$$

which is equal to $\varphi_{\infty,+}(x_1)\varphi_{\infty,-}(x_2)$ for the Gaussians on $V_+$ and $U$, respectively. We also have $\omega(g'_\tau)\varphi_\infty = \omega_+(g'_\tau)\varphi_{\infty,+} \otimes \omega_-(g'_\tau)\varphi_{\infty,-}$ for the corresponding Weil representations. For this decomposition of $V$, we can write the theta function on $S(V(\mathbb{A}_f))$ as a tensor product of two distributions, one on $S(V_+(\mathbb{A}_f))$ and one on $S(U(\mathbb{A}_f))$. To see this, let $\varphi \in S(V(\mathbb{A}_f))$. The theta functions are linear, so it suffices to look at a factorizable Schwartz function $\varphi = \varphi_+ \otimes \varphi_-$. This gives

$$\begin{aligned}
\theta(\tau, z_0, h; \varphi) &= v^{-\frac{l}{2}} \sum_{x \in V(\mathbb{Q})} \omega(g'_\tau)(\varphi_\infty(\cdot, z_0) \otimes \omega(h)\varphi)(x) \\
&= v^{-\frac{l}{2}} \sum_{x_1,x_2} (\omega_+(g'_\tau)\varphi_{\infty,+}(x_1)\varphi_+(h_+^{-1}x_1))(\omega_-(g'_\tau)\varphi_{\infty,-}(x_2)\varphi_-(h_-^{-1}x_2)) \\
&= v^{-\frac{n}{4}} \left( \sum_{x_1} \omega_+(g'_\tau)\varphi_{\infty,+}(x_1)\varphi_+(h_+^{-1}x_1) \right) \\
&\quad \times\ v^{\frac{1}{2}} \left( \sum_{x_2} \omega_-(g'_\tau)\varphi_{\infty,-}(x_2)\varphi_-(h_-^{-1}x_2) \right) \\
&= \theta_+(\tau, z_0, h_+; \varphi_+)\, \theta_-(\tau, z_0, h_-; \varphi_-).
\end{aligned}$$

Hence,

$$\theta(\tau, z_0, h) = \theta_+(\tau, z_0, h_+) \otimes \theta_-(\tau, z_0, h_-),$$

where their respective weights are $\frac{n}{2}$ and $-1$. Since $z_0$ is fixed, we write

$$\theta_\pm(\tau, h_\pm) = \theta_\pm(\tau, z_0, h_\pm).$$

3.2. **The Contraction Map.** Now we describe the main way in which we use the above factorization of the theta function. Let $\varphi \in S(V(\mathbb{A}_f))$. Then we can write $\varphi = \sum_j \varphi_+^j \otimes \varphi_-^j$, where $\varphi_+^j \in S(V_+(\mathbb{A}_f)), \varphi_-^j \in S(U(\mathbb{A}_f))$ and the sum is finite. We define the *contraction map*

$$\langle \cdot, \theta_+(\tau, h_+) \rangle : S(V(\mathbb{A}_f)) \to S(U(\mathbb{A}_f))$$

by

$$\langle \varphi, \theta_+(\tau, h_+) \rangle := \sum_j \theta_+(\tau, h_+; \varphi_+^j)\, \varphi_-^j.$$

It is clear that

$$(( \varphi, \theta(\tau, z_0, h) )) = (( \langle \varphi, \theta_+(\tau, h_+) \rangle, \theta_-(\tau, h_-) )). \tag{34}$$

The expression on the right hand side is nice because it is the pairing of a function in $S(U(\mathbb{A}_f))$ and the theta function for $U$. This is just as in the $n = 0$ case. The value of the contraction map that we are interested in is $\langle F(\tau), \theta_+(\tau, 1) \rangle$.

**Proposition 3.1.** *If $F : \mathfrak{H} \to S_L$ is a weakly holomorphic modular form of weight $1 - \frac{n}{2}$ and type $\omega$ for $\Gamma'$ whose non-positive Fourier coefficients lie in $\mathbb{Z}$, then*
*(i) $\langle F(\tau), \theta_+(\tau, 1)\rangle$ is a weakly holomorphic modular form of weight 1 and type $\omega_-$ for $\Gamma'$ (cf. Definition 2.1),*
*(ii) $\langle F(\tau), \theta_+(\tau, 1)\rangle \in S_{L_-}$ for $L_- = U \cap L$,*
*(iii) The non-positive Fourier coefficients of $\langle F(\tau), \theta_+(\tau, 1)\rangle$ lie in $\mathbb{Z}$.*

*Proof.* By definition,

$$\langle F(\gamma'\tau), \theta_+(\gamma'\tau, h_+)\rangle = (c\tau + d) \left\langle \omega(\gamma')(F(\tau)), \omega_+^\vee(\gamma')(\theta_+(\tau, h_+))\right\rangle_U . \tag{35}$$

Assume that $F(\tau) = \sum_j \varphi_+^j \otimes \varphi_-^j$. We have

$$\omega_+^\vee(\gamma')(\theta_+(\tau, h_+)) = \theta_+(\tau, h_+; \omega_+(\gamma')^{-1} \circ \cdot),$$

so (35) is

$$\begin{aligned}
&= (c\tau + d) \left\langle \sum_j \omega_+(\gamma')(\varphi_+^j) \otimes \omega_-(\gamma')(\varphi_-^j), \theta_+(\tau, h_+; \omega_+(\gamma')^{-1} \circ \cdot) \right\rangle_U \\
&= (c\tau + d) \sum_j \theta_+(\tau, h_+; \omega_+(\gamma')^{-1}\omega_+(\gamma')(\varphi_+^j))\omega_-(\gamma')(\varphi_-^j) \\
&= (c\tau + d) \sum_j \theta_+(\tau, h_+; \varphi_+^j)\omega_-(\gamma')(\varphi_-^j) \\
&= (c\tau + d)\omega_-(\gamma') \left(\langle F(\tau), \theta_+(\tau, h_+)\rangle\right).
\end{aligned}$$

This proves (i).

In order to compute the Fourier expansion of $\langle F(\tau), \theta_+(\tau, h_+)\rangle$, we need the expansion of $\theta_+(\tau, h_+; \varphi_+)$ for $\varphi_+ \in S(V_+(\mathbb{A}_f))$. We take $h_+ = 1$ since the integral we are interested in is

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \Phi(z_0, h; F)dh.$$

The explicit $\mathbf{q}$-expansion of $\theta_+(\tau, 1; \varphi_+)$ is obtained via the action of the Weil representation on $S(V_+(\mathbb{R}))$. In our particular case,

$$\begin{aligned}
\theta_+(\tau, 1; \varphi_+) &= v^{-\frac{n}{4}} \sum_{x_1 \in V_+(\mathbb{Q})} \omega_+(g'_\tau)\varphi_{\infty,+}(x_1)\varphi_+(x_1) \\
&= v^{-\frac{n}{4}} \sum_{x_1} \omega_+(g'_\tau)e^{-\pi(x_1,x_1)}\varphi_+(x_1),
\end{aligned}$$

which by [16] is

$$\begin{aligned}
&= v^{-\frac{n}{4}} \sum_{x_1} v^{\frac{n}{4}} e^{2\pi i u Q(x_1)} e^{-\pi v(x_1,x_1)}\varphi_+(x_1) \\
&= \sum_{x_1} e^{2\pi i \tau Q(x_1)}\varphi_+(x_1) \\
&= \sum_{m\in\mathbb{Q}} \Big( \sum_{\substack{x_1 \\ Q(x_1)=m}} \varphi_+(x_1)\Big)\mathbf{q}^m. \qquad (36)
\end{aligned}$$

Define

$$d_{\varphi_+}(m) := \sum_{\substack{x_1 \\ Q(x_1)=m}} \varphi_+(x_1).$$

Let $L_+ \subset V_+$ be a lattice. Note that if $\varphi_+$ is the characteristic function of a coset $\lambda_+ + L_+$, then $d_{\varphi_+}(m)$ is an integer which counts the number of vectors $x_1 \in \lambda_+ + L_+$ such that $Q(x_1) = m$. Also, $V_+(\mathbb{Q})$ is positive definite so $m \geq 0$ in (36).

Now we compute the Fourier expansion of $\langle F(\tau), \theta_+(\tau, 1)\rangle$. We know $F(\tau) \in S_L$ for some lattice $L \subset V$. If we let $L_+ = V_+ \cap L$ and $L_- = U \cap L$, then generally the lattice $L$ does not split, i.e., $L \supsetneq L_+ + L_-$. We have

$$L_+ + L_- \subset L \subset L^\vee \subset L_+^\vee + L_-^\vee.$$

Let

$$L^\vee = \bigcup_\eta (\eta + L),\ L = \bigcup_\lambda (\lambda + L_+ + L_-),$$

where $\eta$ and $\lambda$ range over $L^\vee/L$ and $L/(L_+ + L_-)$, respectively. If we write $\eta = \eta_+ + \eta_-$ and $\lambda = \lambda_+ + \lambda_-$, then

$$L^\vee = \bigcup_\eta \bigcup_\lambda (\eta_+ + \lambda_+ + L_+) + (\eta_- + \lambda_- + L_-)\,.$$

Let $F(\tau) = \sum_\eta F_\eta(\tau)\varphi_{\eta+L}$ for $\varphi_{\eta+L} = \mathrm{char}(\eta + L)$. Then

$$\varphi_{\eta+L} = \sum_\lambda \varphi_{\eta_+ + \lambda_+ + L_+} \otimes \varphi_{\eta_- + \lambda_- + L_-},$$

and we have

$$F(\tau) = \sum_\eta F_\eta(\tau) \sum_\lambda \left(\varphi_{\eta_+ + \lambda_+ + L_+} \otimes \varphi_{\eta_- + \lambda_- + L_-}\right).$$

By definition of the contraction map, this gives

$$\langle F(\tau), \theta_+(\tau, 1)\rangle = \sum_\eta \sum_\lambda F_\eta(\tau)\theta_+\left(\tau, 1; \varphi_{\eta_+ + \lambda_+ + L_+}\right)\varphi_{\eta_- + \lambda_- + L_-}. \tag{37}$$

From (37), we see that

$$\langle F(\tau), \theta_+(\tau, 1)\rangle \in S_{L_-},$$

but we point out that the cosets $\eta_- + \lambda_- + L_-$ need not be incongruent mod $L_-$. Let $c_\eta(m) = c_{\varphi_{\eta+L}}(m)$ and $d_{\eta_+ + \lambda_+}(m) = d_{\varphi_{\eta_+ + \lambda_+ + L_+}}(m)$. Then the Fourier expansion of $\langle F(\tau), \theta_+(\tau, 1)\rangle$ is

$$\begin{aligned}\langle F(\tau), \theta_+(\tau, 1)\rangle &= \sum_\eta \sum_\lambda \Big(\sum_m c_\eta(m)\mathbf{q}^m\Big)\Big(\sum_m d_{\eta_+ + \lambda_+}(m)\mathbf{q}^m\Big)\varphi_{\eta_- + \lambda_- + L_-} \\ &= \sum_\eta \sum_\lambda \sum_m \Big(\sum_{m_1 + m_2 = m} c_\eta(m_1)d_{\eta_+ + \lambda_+}(m_2)\Big)\mathbf{q}^m \varphi_{\eta_- + \lambda_- + L_-} \\ &= \sum_\eta \sum_\lambda \sum_m C_{\eta, \lambda_+}(m)\mathbf{q}^m \varphi_{\eta_- + \lambda_- + L_-},\end{aligned}$$

where we define

$$C_{\eta, \lambda_+}(m) := \sum_{m_1 + m_2 = m} c_\eta(m_1)d_{\eta_+ + \lambda_+}(m_2).$$

The coefficients $d_{\eta_+ + \lambda_+}(m) \in \mathbb{Z}_{\geq 0}$ for $m \geq 0$ and $d_{\eta_+ + \lambda_+}(m) = 0$ if $m < 0$. So assuming $c_\eta(m) \in \mathbb{Z}$ for $m \leq 0$ implies $C_{\eta, \lambda_+}(m) \in \mathbb{Z}$ for $m \leq 0$, and this finishes the proof of Proposition 3.1. □

We have seen that the zeroth Fourier coefficient of the modular form $F$ is very important. For example, it gives the weight of $\Psi(F)^2$. When doing the general case, we use the contraction map to go from a modular form $F \in S(V(\mathbb{A}_f))$ to $\langle F, \theta_+\rangle \in S(U(\mathbb{A}_f))$. Hence, we will want to know the zeroth coefficient of $\langle F, \theta_+\rangle$. For any modular form $\tilde{F} \in S(U(\mathbb{A}_f))$, define

$$c_0(0)(\tilde{F})$$

to be the zeroth Fourier coefficient of $\tilde{F}$.

**Corollary 3.2.** *The Fourier expansion of $\langle F(\tau), \theta_+(\tau, 1)\rangle$ is*

$$\langle F(\tau), \theta_+(\tau, 1)\rangle = \sum_\eta \sum_\lambda \sum_m C_{\eta,\lambda_+}(m)\mathbf{q}^m \varphi_{\eta_-+\lambda_-+L_-},$$

*where*

$$C_{\eta,\lambda_+}(m) = \sum_{m_1+m_2=m} c_\eta(m_1) d_{\eta_++\lambda_+}(m_2),$$

*and*

$$c_0(0)(\langle F, \theta_+\rangle) := c_0(0)(\langle F(\tau), \theta_+(\tau, 1)\rangle) = \sum_\eta \sum_{\substack{\lambda \\ \eta_-+\lambda_-=0}} C_{\eta,\lambda_+}(0). \tag{38}$$

3.3. **The $(n, 2)$-Theorem.** Recall that the lattice $L$ may not split. For $\eta \in L^\vee/L$ and $\lambda \in L/(L_+ + L_-)$ we write $\eta = \eta_+ + \eta_-$ and $\lambda = \lambda_+ + \lambda_-$.

**Definition 3.3.** *Define*

$$\kappa_\eta(m) := \sum_\lambda \sum_{x \in \eta_++\lambda_++L_+} \kappa_{\eta_-+\lambda_-}(m - Q(x)).$$

Note that Definition 2.17 implies the sum over $x \in \eta_+ + \lambda_+ + L_+$ is finite.

**Theorem 3.4** (The $(n, 2)$-Theorem)**.** *Let $F : \mathfrak{H} \to S_L \subset S(V(\mathbb{A}_f))$ be a weakly holomorphic modular form for $\Gamma'$ of weight $1 - \frac{n}{2}$, with Fourier expansion*

$$F(\tau) = \sum_\eta F_\eta(\tau)\varphi_\eta = \sum_\eta \sum_m c_\eta(m)\mathbf{q}^m \varphi_\eta, \tag{39}$$

*where $\varphi_\eta = \mathrm{char}(\eta + L)$ and $\eta$ runs over $L^\vee/L$. Also, assume $c_\eta(m) \in \mathbb{Z}$ for $m \le 0$. Define*

$$\Phi(z, h; F) := \int^\bullet_{\Gamma\backslash\mathfrak{H}} ((\ F(\tau), \theta(\tau, z, h)\ ))d\mu(\tau).$$

*For $z_0 \in D_0$ we have*

*(i) $\Phi(z_0, h; F)$ is always finite,*

*(ii)* $$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \Phi(z_0, h; F)dh = 2\sum_\eta \sum_{m\ge 0} c_\eta(-m)\kappa_\eta(m).$$

*Proof.* The regularized integral is given by

$$\Phi(z_0, h; F) = \int^\bullet_{\Gamma\backslash\mathfrak{H}} \phi(\tau)d\mu(\tau),$$

where the integrand is

$$\begin{aligned}\phi(\tau) &= ((\ F(\tau), \theta(\tau, z_0, h)\ )) \\ &= ((\ \langle F(\tau), \theta_+(\tau, 1)\rangle, \theta_-(\tau, h_-)\ )),\end{aligned}$$

as in (34). Hence,

$$\Phi(z_0, h; F) = \Phi(z_0, h_-; \langle F(\tau), \theta_+(\tau, 1)\rangle), \tag{40}$$

and Proposition 2.6 implies (40) is always finite. We remark that the regularization process does not depend on the integrand $\phi(\tau)$.

For (ii), using (40) the desired integral can be written

$$\begin{aligned}
&\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \Phi(z_0, h; F)dh \\
&= \int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \Phi(z_0, h_-; \langle F(\tau), \theta_+(\tau, 1)\rangle)dh_-. \qquad (41)
\end{aligned}$$

Proposition 3.1 tells us we may apply the $(0, 2)$-Theorem to (41). Doing this we see

$$\begin{aligned}
(41) &= 2\sum_\eta \sum_\lambda \sum_{m\geq 0} C_{\eta,\lambda_+}(-m)\kappa_{\eta_-+\lambda_-}(m) \\
&= 2\sum_\eta \sum_\lambda \sum_{m\geq 0} \Big( \sum_{m_1+m_2=-m} c_\eta(m_1)d_{\eta_++\lambda_+}(m_2)\Big)\kappa_{\eta_-+\lambda_-}(m) \\
&= 2\sum_\eta \sum_\lambda \sum_{m\geq 0} \Big( \sum_{m_1\leq 0} c_\eta(m_1)d_{\eta_++\lambda_+}(-m-m_1)\Big)\kappa_{\eta_-+\lambda_-}(m) \\
&= 2\sum_\eta \sum_\lambda \sum_{m\geq 0} \Big( \sum_{m_1\geq 0} c_\eta(-m_1)d_{\eta_++\lambda_+}(m_1-m)\Big)\kappa_{\eta_-+\lambda_-}(m). \qquad (42)
\end{aligned}$$

If $m > m_1$, then $d_{\eta_++\lambda_+}(m_1 - m) = 0$, so

$$(42) = 2\sum_\eta \sum_\lambda \sum_{m_1\geq 0} c_\eta(-m_1)\Big( \sum_{0\leq m\leq m_1} d_{\eta_++\lambda_+}(m_1-m)\kappa_{\eta_-+\lambda_-}(m)\Big).$$

Then

$$\begin{aligned}
&\sum_{0\leq m\leq m_1} d_{\eta_++\lambda_+}(m_1-m)\kappa_{\eta_-+\lambda_-}(m) \\
&= \sum_{0\leq m\leq m_1} (\#\{x\in \eta_+ + \lambda_+ + L_+ \mid Q(x) = m_1 - m\})\, \kappa_{\eta_-+\lambda_-}(m) \\
&= \sum_{\substack{x\in\eta_++\lambda_++L_+ \\ 0\leq Q(x)\leq m_1}} \kappa_{\eta_-+\lambda_-}(m_1 - Q(x)) \\
&= \sum_{x\in\eta_++\lambda_++L_+} \kappa_{\eta_-+\lambda_-}(m_1 - Q(x)),
\end{aligned}$$

since $Q(x) \geq 0$ and $\kappa_{\eta_-+\lambda_-}(m) = 0$ for $m < 0$. So

$$(42) = 2\sum_\eta \sum_{m\geq 0} c_\eta(-m)\kappa_\eta(m).$$

□

We now state an important corollary of Theorem 3.4, which gives the average value of the logarithm of a Borcherds form over CM points. As in section 2.3, let $T = \mathrm{GSpin}(U)$ and let $K \subset H(\mathbb{A}_f)$ be a compact open subgroup such that

$F : \mathfrak{H} \to S_L^K$. Write $K_T = K \cap T(\mathbb{A}_f)$ and recall that we consider the set of CM points

$$Z(U)_K = T(\mathbb{Q})\backslash\Big(D_0 \times T(\mathbb{A}_f)/K_T\Big) \hookrightarrow X_K.$$

**Corollary 3.5.** *(i) When $z_0$ is not in the divisor of the Borcherds form $\Psi(F)$ (i.e., when (11) holds), the result of Theorem 3.4 can be stated as*

$$\sum_{z\in Z(U)_K} \log||\Psi(z;F)||^2 = \frac{-2}{\mathrm{vol}(K_T)}\Big(\sum_{\eta}\sum_{m\geq 0} c_\eta(-m)\kappa_\eta(m)\Big).$$

*(ii) If $U \simeq k = \mathbb{Q}(\sqrt{-d})$ where $-d$ is an odd fundamental discriminant, then we have the factorization*

$$\prod_{z\in Z(U)_K} ||\Psi(z;F)||^2 = \mathbf{rat}\cdot\left((4d\pi)^{-1}e^{2\frac{L'(0,\chi_d)}{L(0,\chi_d)}}\right)^{h_k c_0(0)(\langle F,\theta_+\rangle)},$$

*where $\mathbf{rat} \in \mathbb{Q}$ and $c_0(0)(\langle F,\theta_+\rangle)$ is the zeroth Fourier coefficient of $\langle F,\theta_+\rangle$, as defined in (38). Note that the degree of $Z(U)_K$ is $2h_k$, where $h_k$ is the class number of $k$. This factorization can also be written as*

$$\prod_{z\in Z(U)_K} ||\Psi(z;F)||^2 = \mathbf{rat}\cdot\left[(4d\pi)^{-h_k}\prod_{a=1}^{d-1}\Gamma\left(\frac{a}{d}\right)^{w_k\chi_d(a)}\right]^{c_0(0)(\langle F,\theta_+\rangle)},$$

*where $w_k$ is the number of roots of unity in $k$. The transcendental factor appearing in this factorization is related to Shimura's period invariants* [15],[8],[18].
*(iii)*

$$\log(\mathbf{rat}) = -h_k\sum_{\eta}\sum_{m>0} c_\eta(-m)\Big(\sum_{\lambda}\sum_{\substack{x\in\eta_++\lambda_++L_+\\ Q(x)<m}} \kappa_{\eta_-+\lambda_-}(m-Q(x))\Big).$$

*Proof.* (i) follows from (11). For (ii) and (iii) we have $\mathrm{vol}(K_T) = \frac{2}{h_k}$, and we will see from Theorem 4.1 of the next section that

$$-h_k\sum_{\eta}\sum_{m>0} c_\eta(-m)\Big(\sum_{\lambda}\sum_{\substack{x\in\eta_++\lambda_++L_+\\ Q(x)<m}} \kappa_{\eta_-+\lambda_-}(m-Q(x))\Big) \tag{43}$$

is the logarithm of a rational number $\mathbf{rat}$. From $\Lambda(s,\chi_d) = \pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)L(s,\chi_d)$, we see

$$\frac{\Lambda'(1,\chi_d)}{\Lambda(1,\chi_d)} = -\frac{1}{2}\log(\pi) + \Gamma'(1) + \frac{L'(1,\chi_d)}{L(1,\chi_d)}.$$

So for the corresponding part of (43) that involves $\kappa_0(0)$, we have

$$-h_k c_0(0)(\langle F,\theta_+\rangle)\left(\log(d) - \log(\pi) + 2\Gamma'(1) + 2\frac{L'(1,\chi_d)}{L(1,\chi_d)}\right),$$

which equals

$$h_k c_0(0)(\langle F,\theta_+\rangle)\left(2\frac{L'(0,\chi_d)}{L(0,\chi_d)} - \log(4d\pi)\right).$$

The second identity in (ii) follows from the Chowla-Selberg formula (cf. Proposition 10.10 of [14]), which says

$$\frac{L'(0,\chi_d)}{L(0,\chi_d)} = \frac{w_k}{2h_k}\sum_{a=1}^{d-1}\chi_d(a)\log\Gamma\left(\frac{a}{d}\right).$$

□

As an immediate consequence of Corollary 3.5 and Theorem 4.1 of the next section, we obtain a Gross-Zagier phenomenon about which primes can occur in the factorization of the rational part of

$$\prod_{z\in Z(U)_K} ||\Psi(z;F)||^2.$$

For $F$ as in (39), define

$$m_{\max} = \max\{m > 0 \mid c_\eta(-m) \neq 0 \text{ for some } \eta\}.$$

**Theorem 3.6.** *Let $-d$ be an odd fundamental discriminant and assume $U \simeq k = \mathbb{Q}(\sqrt{-d})$. Then the only primes which occur in the factorization of the rational part of*

$$\prod_{z\in Z(U)_K} ||\Psi(z;F)||^2$$

*are*

*(i) $q$ such that $q \mid d$,*

*(ii) $p$ inert in $k$ with $p \le dm_{\max}$.*

Note that this fact holds for all Borcherds forms and all CM points. In addition, we point out that the modular form $F$ is not needed in order to obtain $m_{\max}$. It can be recovered from the divisor of $\Psi(F)^2$ (cf. Theorem 1.3 of [12]).

## 4. Explicit computation of $\kappa_\mu(t)$ for $t \in \mathbb{Q}_{>0}$

In order to compute examples of our main theorem, we need to derive explicit formulas for $\kappa_\mu(t)$ for $t \in \mathbb{Q}_{>0}$. Our previous discussion of the Clifford algebra of $U$ and Lemma 2.12 imply that, without loss of generality, we may assume $U = k$ is an imaginary quadratic field with quadratic form $Q$ given by a negative multiple of the norm-form. In this section we assume that $L = \mathfrak{a} \subset \mathcal{O}_k$ is an integral ideal and that $Q(x) = -\frac{Nx}{N\mathfrak{a}}$, so that $L^\vee = \mathcal{D}^{-1}\mathfrak{a}$, where $\mathcal{D}$ is the different of $k$. This is certainly not the most general possible lattice. Write $\kappa_\mu(t)$ as $\kappa(t,\mu,\mathfrak{a})$ for $\mu \in \mathcal{D}^{-1}\mathfrak{a}/\mathfrak{a}$. For simplicity, we assume that $k = \mathbb{Q}(\sqrt{-d})$, where $d > 3, d \equiv 3 \pmod 4$ and is square-free, so that the prime 2 is not ramified. Let $\chi$ be the character of $\mathbb{Q}_{\mathbb{A}}^\times$ associated to $k$, which is defined via the global quadratic Hilbert symbol so that $\chi(t) = (t,-d)_{\mathbb{A}}$. Then for a prime $p \le \infty$, the local character is $\chi_p(t) = (t,-d)_p$ where $(\ ,\ )_p$ is the local quadratic Hilbert symbol.

Throughout this section we let $p$ denote an unramified prime and $q$ denote a ramified prime. Let $\mu$ be a coset in $\mathcal{D}^{-1}\mathfrak{a}/\mathfrak{a}$. Write $\mu_q$ for the image of $\mu$ under the map

$$\mathcal{D}^{-1}\mathfrak{a}/\mathfrak{a} \to \mathcal{D}^{-1}\mathfrak{a}_q/\mathfrak{a}_q,$$

where $\mathfrak{a}_q = \mathfrak{a} \otimes_{\mathbb{Z}} \mathbb{Z}_q$. For $t \in \mathbb{Q}_{>0}$, we introduce the function

$$\rho(t) = \#\{\mathfrak{a} \subseteq \mathcal{O}_k \mid N\mathfrak{a} = t\}.$$

This function factors as

$$\rho(t) = \prod_p \rho_p(t), \tag{44}$$

where $\rho_p(t) = \rho(p^{\mathrm{ord}_p(t)})$. The explicit formula for $\kappa(t, \mu, \mathfrak{a})$ is given by the following theorem.

**Theorem 4.1.** *For $\mu \in \mathcal{D}^{-1}\mathfrak{a}/\mathfrak{a}$ and $t \in \mathbb{Q}_{>0}$,*

$$\kappa(t, \mu, \mathfrak{a}) = -\frac{1}{h_k} \prod_{q|d} \mathrm{char}(Q(\mu_q) + \mathbb{Z}_q)(t) \times$$

$$\left[\rho(dt) \sum_{\substack{q|d \\ \mu_q=0}} \eta_q(t,\mu)(\mathrm{ord}_q(t)+1)\log(q) + \eta_0(t,\mu) \sum_{p \text{ inert}} (\mathrm{ord}_p(t)+1)\rho\left(\frac{dt}{p}\right)\log(p)\right],$$

*where*

$$\eta_q(t,\mu) = (1 - \chi_q(-t)) \prod_{\substack{q'|d \\ q' \neq q \\ \mu_{q'}=0}} (1 + \chi_{q'}(-t))$$

*and*

$$\eta_0(t,\mu) = \prod_{\substack{q|d \\ \mu_q=0}} (1 + \chi_q(-t)).$$

*We take $\eta_0(t,\mu) = 1$ if $\mu_q \neq 0$ for all $q \mid d$. For $t = 0$,*

$$\kappa(0, 0, \mathfrak{a}) = \log(d) + 2\frac{\Lambda'(1, \chi_d)}{\Lambda(1, \chi_d)}.$$

Note that when $\mu_q \neq 0$ for all $q \mid d$ we have $\eta_q(t,\mu) = 0$ for all $q$ and $\eta_0(t,\mu) = 1$, and so we get a much simpler formula in this "generic" case.

*Proof.* The value for $t = 0$ is defined in Definition 2.17. For $t > 0$, $\kappa(t, \mu, \mathfrak{a})$ is given by the second term in the Laurent expansion of a certain Eisenstein series. These Eisenstein series have factorizations in terms of local Whittaker functions, and we use these factorizations to derive the above formula for $\kappa(t, \mu, \mathfrak{a})$. Let $\varphi_{\mu_q}$ be the characteristic function of the coset $\mu_q$, $X = p^{-s}$, and $\tau = u + iv \in \mathfrak{H}$. Using [17] and [13], we have the following formulas for the normalized local Whittaker functions. For $\mu = 0$, Lemma 2.3 of [13] tells us we only need to consider $t \in \mathbb{Z}$, and for $t > 0$ we have,

$$W^*_{t,\infty}(\tau, s) = \gamma_\infty v^{\frac{1-s}{2}} e(tu) \frac{2i\pi^{\frac{s}{2}} e^{2\pi tv}}{\Gamma(\frac{s}{2})} \int_{u>2tv} e^{-2\pi u} u^{\frac{s}{2}} (u - 2tv)^{\frac{s}{2}-1} du, \tag{45}$$

$$W^*_{t,p}(s, \varphi_0) = \sum_{r=0}^{\mathrm{ord}_p(t)} (\chi_p(p)X)^r, \tag{46}$$

$$W^*_{t,q}(s, \varphi_0) = \gamma_q q^{-\frac{1}{2}} \begin{cases} 1 + (q, -t)_q X^{\mathrm{ord}_q(t)+1} & \text{if } \mathrm{ord}_q(t) \text{ is even,} \\ 1 + (q, -dt)_q X^{\mathrm{ord}_q(t)+1} & \text{if } \mathrm{ord}_q(t) \text{ is odd.} \end{cases} \tag{47}$$

Here $\gamma_\infty$ and $\gamma_q$ are local factors which do not affect our global computations since $\gamma_\infty \prod_q \gamma_q = 1$, where the product is over all ramified primes. For an unramified prime $p$, the local lattice $\mathfrak{a}_p = \mathfrak{a} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is unimodular. Note that here is where we have lost generality by assuming $L = \mathfrak{a}$ is an integral ideal. Since $\mathfrak{a}_p$ is unimodular,

we only need to consider the Whittaker functions for nonzero cosets at ramified primes. For $\mu_q \neq 0$ we have

$$W^*_{t,q}(s, \varphi_{\mu_q}) = \gamma_q q^{-\frac{1}{2}} \mathrm{char}(Q(\mu_q) + \mathbb{Z}_q)(t). \tag{48}$$

Note that in (48), $W^*_{t,q}(s, \varphi_{\mu_q})$ is either a nonzero constant or is identically zero. Following [13], the normalized Eisenstein series has Fourier coefficients given by

$$E^*_t(\tau, s, \Phi^{1,\mu}) = v^{-\frac{1}{2}} d^{\frac{s+1}{2}} W^*_{t,\infty}(\tau, s) \prod_{q|d} W^*_{t,q}(s, \varphi_\mu) \prod_{p\nmid d} W^*_{t,p}(s, \varphi_0). \tag{49}$$

Write $t = q^{\alpha_q} u$ where $\alpha_q = \mathrm{ord}_q(t)$. We now show that (47) can be combined into one nice formula.

**Lemma 4.2.** $W^*_{t,q}(s, \varphi_0) = \gamma_q q^{-\frac{1}{2}} \left(1 + \chi_q(-t) X^{\alpha_q+1}\right)$.

*Proof.* For $\alpha_q$ even, we have

$$(q, -t)_q = (-t, q)_q = (-t, -1)_q(-t, -q)_q = (-t, -1)_q(-t, dq)_q \chi_q(-t),$$

and

$$(-t, -1)_q(-t, dq)_q = (-t, -dq^{-1})_q = \left(\frac{-dq^{-1}}{q}\right)^{\alpha_q} = 1.$$

For $\alpha_q$ odd,

$$\begin{aligned}(q, -dt)_q &= (-1)^{\frac{q-1}{2}}(q, d)_q(q, t)_q \\ &= (-1, q)_q(q, d)_q(-t, -q)_q(-1, q)_q(-t, -1)_q \\ &= (q, d)_q(-t, -1)_q(-t, dq)_q \chi_q(-t),\end{aligned}$$

and

$$\begin{aligned}(q, d)_q(-t, -1)_q(-t, dq)_q &= (q, d)_q(-t, -dq^{-1})_q \\ &= (-1)^{\frac{q-1}{2}} \left(\frac{dq^{-1}}{q}\right)^{\alpha_q} \left(\frac{-dq^{-1}}{q}\right)^{\alpha_q} \\ &= 1.\end{aligned}$$

So (47) can be rewritten as

$$W^*_{t,q}(s, \varphi_0) = \gamma_q q^{-\frac{1}{2}} \left(1 + \chi_q(-t) X^{\alpha_q+1}\right). \tag{50}$$

□

Let us first compute $\kappa(t, \mu, \mathfrak{a})$ for $\mu = 0$ and $t \in \mathbb{N}$. To do this, we need the following special values for the local Whittaker functions, cf. Lemma 2.5 and Propositions 2.6 and 2.7 of [13].

**Lemma 4.3.** *At $s = 0$ we have*
*(i)* $W^*_{t,\infty}(\tau, 0) = -\gamma_\infty 2v^{\frac{1}{2}} e(t\tau)$.
*(ii)* $W^*_{t,p}(0, \varphi_0) = \rho_p(t)$, *and if* $\rho_p(t) = 0$ *then*

$$W^{*,\prime}_{t,p}(0, \varphi_0) = \frac{1}{2}(\mathrm{ord}_p(t) + 1)\rho_p\left(\frac{t}{p}\right)\log(p).$$

*(iii)* $W^*_{t,q}(0, \varphi_0) = \gamma_q q^{-\frac{1}{2}}(1 + \chi_q(-t))$, *and if* $\chi_q(-t) = -1$ *then*

$$W^{*,\prime}_{t,q}(0, \varphi_0) = \gamma_q q^{-\frac{1}{2}}(\mathrm{ord}_q(t) + 1)\rho_q(t)\log(q).$$

Given (49), we consider different cases based on when one and only one local Whittaker function vanishes at $s=0$. Since $W^*_{t,\infty}(\tau,0)\neq 0$ for $t\in\mathbb{N}$, there are two cases.

<u>Case 1</u>: $W^*_{t,p}(0,\varphi_0)=0$ for $p$ unramified, $W^*_{t,p'}(0,\varphi_0)\neq 0$ $\forall p'\neq p$.

$W^*_{t,p}(0,\varphi_0)=0$ implies that $p$ is inert and $\mathrm{ord}_p(t)$ is odd. Since $W^*_{t,q}(0,\varphi_0)\neq 0$ for $q$ ramified, we have $\chi_q(-t)=1$ and $W^*_{t,q}(0,\varphi_0)=\gamma_q 2q^{-\frac{1}{2}}$. Computing the derivative of the Fourier coefficient we get

$$E_t^{*,\prime}\big(\tau,0,\Phi^{1,0}\big)=W^{*,\prime}_{t,p}(0,\varphi_0)\left[v^{-\frac{1}{2}}d^{\frac{1}{2}}W^*_{t,\infty}(\tau,0)\prod_{q|d}W^*_{t,q}(0,\varphi_0)\prod_{\substack{p'\nmid d\\ p'\neq p}}W^*_{t,p'}(0,\varphi_0)\right]$$

$$=\log(p)\frac{1}{2}(\mathrm{ord}_p(t)+1)\rho_p\left(\frac{t}{p}\right)\left[-v^{-\frac{1}{2}}d^{\frac{1}{2}}\gamma_\infty 2v^{\frac{1}{2}}e(t\tau)2^{\nu(d)}\prod_{q|d}\gamma_q q^{-\frac{1}{2}}\prod_{\substack{p'\nmid d\\ p'\neq p}}\rho_{p'}(t)\right]$$

$$=-\log(p)(\mathrm{ord}_p(t)+1)\rho_p\left(\frac{t}{p}\right)e(t\tau)2^{\nu(d)}\prod_{q|d}\rho_q\left(\frac{t}{p}\right)\prod_{\substack{p'\nmid d\\ p'\neq p}}\rho_{p'}\left(\frac{t}{p}\right),$$

since $\rho_q\left(\frac{t}{p}\right)=1$ and $\rho_{p'}(t)=\rho_{p'}\left(\frac{t}{p}\right)$, and where $\nu(d)$ is the number of primes dividing $d$. So we see

$$E_t^{*,\prime}\big(\tau,0,\Phi^{1,0}\big)=-\log(p)(\mathrm{ord}_p(t)+1)2^{\nu(d)}\rho\left(\frac{t}{p}\right)e(t\tau). \tag{51}$$

<u>Case 2</u>: $W^*_{t,q}(0,\varphi_0)=0$ for $q$ ramified, $W^*_{t,p}(0,\varphi_0)\neq 0$ $\forall p\neq q$.

$W^*_{t,q}(0,\varphi_0)=0$ implies $\chi_q(-t)=-1$ while for any ramified prime $q'\neq q$ we have $\chi_{q'}(-t)=1$ and $W^*_{t,q'}(0,\varphi_0)=\gamma_{q'}2(q')^{-\frac{1}{2}}$. In this case, we see

$$\begin{aligned}E_t^{*,\prime}\big(\tau,0,\Phi^{1,0}\big)=&W^{*,\prime}_{t,q}(0,\varphi_0)\left[v^{-\frac{1}{2}}d^{\frac{1}{2}}W^*_{t,\infty}(\tau,0)\prod_{\substack{q'|d\\ q'\neq q}}W^*_{t,q'}(0,\varphi_0)\prod_{p\nmid d}W^*_{t,p}(0,\varphi_0)\right]\\
=&\gamma_q q^{-\frac{1}{2}}\log(q)(\mathrm{ord}_q(t)+1)\rho_q(t)\times\\
&\left[-v^{-\frac{1}{2}}d^{\frac{1}{2}}\gamma_\infty 2v^{\frac{1}{2}}e(t\tau)2^{\nu(d)-1}\prod_{\substack{q'|d\\ q'\neq q}}\gamma_{q'}(q')^{-\frac{1}{2}}\prod_{p\nmid d}\rho_p(t)\right]\\
=&-\log(q)(\mathrm{ord}_q(t)+1)2^{\nu(d)}\rho(t)e(t\tau).\end{aligned}\tag{52}$$

Recall that the definition of $\kappa(t,\mu,\mathfrak{a})$ involves the non-normalized Eisenstein series, and at $s=0$ we have $E^{*,\prime}(\tau,0,\Phi^{1,\mu})=h_kE'(\tau,0,\Phi^{1,\mu})$. This fact and the above analysis, particularly (51) and (52), give

$$\kappa(t,0,\mathfrak{a})=$$

$$-\frac{2^{\nu(d)}}{h_k}\left(\sum_{q|d}\xi_q(t)(\mathrm{ord}_q(t)+1)\rho(t)\log(q)+\sum_{p\text{ inert}}\xi_0(t)(\mathrm{ord}_p(t)+1)\rho\left(\frac{t}{p}\right)\log(p)\right),$$

where

$$\xi_q(t) = \begin{cases} 0 & \text{if } \chi_q(-t) = 1 \text{ or } \chi_q(-t) = -1 = \chi_{q'}(-t), \text{ for some ramified prime} \\ & q' \neq q, \\ 1 & \text{if } \chi_q(-t) = -1 \text{ and } \chi_{q'}(-t) = 1 \text{ for all ramified primes } q' \neq q, \end{cases}$$

and

$$\xi_0(t) = \begin{cases} 0 & \text{if } \chi_q(-t) = -1 \text{ for some ramified prime } q, \\ 1 & \text{otherwise.} \end{cases}$$

Now we compute $\kappa(t, \mu, \mathfrak{a})$ for $\mu \neq 0$. One main thing to keep in mind is that there is at least one ramified prime $q$ such that $\mu_q \neq 0$, but the coset can be zero locally at other ramified primes. Write $\mu = (\mu_p)$, where if $p$ is unramified then $\mu_p = 0$ and let $\alpha(\mu) = \#\{q \text{ ramified} \mid \mu_q = 0\}$. Again, we consider two cases.

Case 1: $W^*_{t,p}(0, \varphi_0) = 0$ for $p$ unramified, $W^*_{t,p'}(0, \varphi_{\mu_{p'}}) \neq 0\ \forall p' \neq p$.

The formula for the derivative of the Fourier coefficient is

$$E_t^{*,\prime}(\tau, 0, \Phi^{1,\mu}) = W^{*,\prime}_{t,p}(0, \varphi_0) \left[ v^{-\frac{1}{2}} d^{\frac{1}{2}} W^*_{t,\infty}(\tau, 0) \prod_{q \mid d} W^*_{t,q}(0, \varphi_{\mu_q}) \prod_{\substack{p' \nmid d \\ p' \neq p}} W^*_{t,p'}(0, \varphi_0) \right].$$

Then after cancelling some terms and using Lemma 4.3 and (48), we get

$$= \log(p) \frac{1}{2} (\mathrm{ord}_p(t) + 1) \rho_p\left(\frac{t}{p}\right) \left[ -2e(t\tau) 2^{\alpha(\mu)} \prod_{\substack{q \mid d \\ \mu_q \neq 0}} \mathrm{char}(Q(\mu_q) + \mathbb{Z}_q)(t) \prod_{\substack{p' \nmid d \\ p' \neq p}} \rho_{p'}(t) \right].$$

If $q$ is a ramified prime with $\mu_q \neq 0$, then $W^*_{t,q}(0, \varphi_{\mu_q}) \neq 0$ implies $\mathrm{ord}_q(t) = -1$. This means $\rho_q(qt) = 1$ and this also equals $\rho_q(dt)$. If $\mu_q = 0$, then $\rho_q(t) = 1 = \rho_q(dt)$. Similarly, $\rho_p\left(\frac{t}{p}\right) = \rho_p\left(\frac{dt}{p}\right)$ and $\rho_{p'}(t) = \rho_{p'}(dt) = \rho_{p'}\left(\frac{dt}{p}\right)$. We also see that if $\mu_q = 0$, then $\mathrm{char}(Q(\mu_q) + \mathbb{Z}_q)(t) = \mathrm{char}(\mathbb{Z}_q)(t) = 1$. So the above formula is

$$= -2^{\alpha(\mu)} \log(p) (\mathrm{ord}_p(t) + 1) \rho\left(\frac{dt}{p}\right) e(t\tau) \prod_{q \mid d} \mathrm{char}(Q(\mu_q) + \mathbb{Z}_q)(t). \tag{53}$$

Case 2: $W^*_{t,q}(0, \varphi_0) = 0$ for $q$ ramified, $W^*_{t,p}(0, \varphi_{\mu_p}) \neq 0\ \forall p \neq q$.

Here the derivative is given by

$$E_t^{*,\prime}(\tau, 0, \Phi^{1,\mu}) = W^{*,\prime}_{t,q}(0, \varphi_0) \left[ v^{-\frac{1}{2}} d^{\frac{1}{2}} W^*_{t,\infty}(\tau, 0) \prod_{\substack{q' \mid d \\ q' \neq q}} W^*_{t,q'}(0, \varphi_{\mu_{q'}}) \prod_{p \nmid d} W^*_{t,p}(0, \varphi_0) \right]$$

$$= \log(q) (\mathrm{ord}_q(t) + 1) \rho_q(t) \left[ -2e(t\tau) 2^{\alpha(\mu) - 1} \prod_{\substack{q \mid d \\ \mu_q \neq 0}} \mathrm{char}(Q(\mu_q) + \mathbb{Z}_q)(t) \prod_{p \nmid d} \rho_p(t) \right]$$

$$= -2^{\alpha(\mu)} \log(q) (\mathrm{ord}_q(t) + 1) \rho(dt) e(t\tau) \prod_{q \mid d} \mathrm{char}(Q(\mu_q) + \mathbb{Z}_q)(t). \tag{54}$$

Note that we do not consider the case where $W^*_{t,q}(0, \varphi_{\mu_q}) = 0$ for $\mu_q \neq 0$, since then the Whittaker function is identically zero and there is no contribution to the derivative. Formulas (53) and (54) imply that for $\mu \neq 0$,

$$\kappa(t, \mu, \mathfrak{a}) = -\frac{2^{\alpha(\mu)}}{h_k} \prod_{q|d} \operatorname{char}(Q(\mu_q) + \mathbb{Z}_q)(t) \times$$

(55)
$$\left( \sum_{q|d} \xi_q(t, \mu)(\operatorname{ord}_q(t) + 1)\rho(dt) \log(q) + \sum_{p \text{ inert}} \xi_0(t, \mu)(\operatorname{ord}_p(t) + 1)\rho\left(\frac{dt}{p}\right) \log(p) \right),$$

where

$$\xi_q(t, \mu) = \begin{cases} 0 & \text{if } \mu_q \neq 0, \text{ or } \mu_q = 0 \text{ and } \chi_q(-t) = 1, \text{ or } \chi_q(-t) = -1 = \chi_{q'}(-t) \\ & \text{for some ramified prime } q' \neq q \text{ with } \mu_{q'} = 0, \\ 1 & \text{if } \mu_q = 0, \chi_q(-t) = -1, \text{ and } \chi_{q'}(-t) = 1 \text{ for all ramified} \\ & \text{primes } q' \neq q \text{ with } \mu_{q'} = 0, \end{cases}$$

and

$$\xi_0(t, \mu) = \begin{cases} 0 & \text{if } \chi_q(-t) = -1 \text{ and } \mu_q = 0 \text{ for some ramified prime } q, \\ 1 & \text{otherwise.} \end{cases}$$

If we take $\mu = 0$ in the above equations, we see that $\xi_q(t, 0) = \xi_q(t)$, $\xi_0(t, 0) = \xi_0(t)$ and $\nu(d) = \alpha(0)$. Also, when $\mu = 0$ then $t \in \mathbb{N}$ so $\rho(dt) = \rho(t), \rho\left(\frac{dt}{p}\right) = \rho\left(\frac{t}{p}\right)$ and the characteristic functions can be ignored. This means (55) holds when $\mu = 0$ as well. We then note that once we sum over $q \mid d$ with $\mu_q = 0$ we can replace $2^{\alpha(\mu)}\xi_q(t, \mu)$ with $\eta_q(t, \mu)$ and we have

$$\eta_0(t, \mu) = 2^{\alpha(\mu)}\xi_0(t, \mu).$$

This finishes the proof of Theorem 4.1. $\square$

## References


1. Scott Ahlgren and Ken Ono, *Arithmetic of singular moduli and class polynomials*, Compos. Math. **141** (2005), no. 2, 293–312. MR MR2134268
2. Richard E. Borcherds, *Automorphic forms with singularities on Grassmannians*, Invent. Math. **132** (1998), no. 3, 491–562.
3. Jan Hendrik Bruinier, *Infinite products in number theory and geometry*, Jahresber. Deutsch. Math.-Verein. **106** (2004), no. 4, 151–184. MR MR2068524
4. Jan Hendrik Bruinier and Jens Funke, *Traces of CM-values of modular functions*, in preparation, 2005.
5. Jan Hendrik Bruinier and Tonghai Yang, *CM values of Hilbert modular functions*, Inventiones math. **163** (2006), 229–288.
6. Claude C. Chevalley, *The algebraic theory of spinors*, Columbia University Press, New York, 1954.
7. Harold Davenport, *Multiplicative number theory*, third ed., Graduate Texts in Mathematics, vol. 74, Springer-Verlag, New York, 2000, Revised and with a preface by Hugh L. Montgomery.
8. Benedict H. Gross, *On the periods of abelian integrals and a formula of Chowla and Selberg*, Invent. Math. **45** (1978), no. 2, 193–211, With an appendix by David E. Rohrlich. MR MR0480542 (58 #701)
9. Benedict H. Gross and Don B. Zagier, *On singular moduli*, J. Reine Angew. Math. **355** (1985), 191–220.

10. Robert Hermann, *Spinors, Clifford and Cayley algebras*, Department of Mathematics, Rutgers University, New Brunswick, N.J., 1974, Interdisciplinary mathematics, Vol. VII.
11. Stephen S. Kudla, *Central derivatives of Eisenstein series and height pairings*, Ann. of Math. (2) **146** (1997), no. 3, 545–646.
12. ______, *Integrals of Borcherds forms*, Compositio Math. **137** (2003), no. 3, 293–349.
13. Stephen S. Kudla, Michael Rapoport, and Tonghai Yang, *On the derivative of an Eisenstein series of weight one*, Internat. Math. Res. Notices (1999), no. 7, 347–385.
14. ______, *Derivatives of Eisenstein series and Faltings heights*, Compos. Math. **140** (2004), no. 4, 887–951. MR **MR2059224 (2005f:**11124)
15. Goro Shimura, *On some arithmetic properties of modular forms of one and several variables*, Ann. of Math. (2) **102** (1975), no. 3, 491–515. MR MR0491519 (58 #10758)
16. Takuro Shintani, *On construction of holomorphic cusp forms of half integral weight*, Nagoya Math. J. **58** (1975), 83–126.
17. Tonghai Yang, preprint, 2001.
18. Hiroyuki Yoshida, *Absolute CM-periods*, Mathematical Surveys and Monographs, vol. 106, American Mathematical Society, Providence, RI, 2003. MR **MR2011848 (2004j:**11057)
19. Don Zagier, *Traces of singular moduli*, Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998), Int. Press Lect. Ser., vol. 3, Int. Press, Somerville, MA, 2002, pp. 211–244.